LECTURES ON STRING THEORY IN CURVED SPACETIMES

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Abstract

Recent progress on string theory in curved spacetimes is reviewed. The string dynamics in cosmological and black hole spacetimes is investigated. The different methods available to solve the string equations of motion and constraints in curved spacetimes are described. That is, the string perturbation approach, the null string approach, the τ-expansion, and the construction of global solutions (for instance by inverse scattering methods).

The classical behaviour of strings in FRW and inflationary spacetimes is now understood in a large extent from the various types of explicit string solutions. Three different types of behaviour appear: **unstable, dual** to unstable and **stable**. For the unstable strings, the energy and size grow for large scale factors $R \to \infty$, proportional to R. For the dual to unstable strings, the energy and size blow up for $R \to 0$ as $1/R$. For stable strings, the energy and proper size are bounded. (In Minkowski spacetime, all string solutions are of the stable type).

Recent progress on self-consistent solutions to the Einstein equations for string dominated universes is reviewed. The energy-momentum tensor for a gas of strings is then considered as source of the spacetime geometry and from the above string behaviours the string equation of state is **derived**. The self-consistent string solution exhibits the realistic matter dominated behaviour $R \simeq T^{2/3}$ for large times and the radiation dominated behaviour $R \simeq T^{1/2}$ for early times (T being the cosmic time).

We report on the **exact integrability** of the string equations plus the constraints in de Sitter spacetime that allows to systematically find **exact** string solutions by soliton methods and the multistring solutions. **Multistring solutions** are a new feature in curved spacetimes. That is, a single world-sheet simultaneously describes many different and independent strings. This phenomenon has no analogue in flat spacetime and follows to the coupling of the strings with the geometry.

Finally, the string dynamics next and inside a Schwarzschild black hole is analyzed and their physical properties discussed.

CONTENTS

I. INTRODUCTION

The construction of a sensible quantum theory of gravitation is probably the greatest challenge in theoretical physics for the end of this century and most probably for the next century too.

Another problem (the most often discussed in connection with gravity quantization) is the one of the renormalizability of the Einstein theory (or its various generalizations) when quantized as a local quantum field theory. Actually, even deeper conceptual problems arise when one tries to combine quantum concepts with General Relativity. For example, statistical phenomena like Hawking's radiation arise when free fields are quantized in black-hole backgrounds. This points out a lack of quantum coherence even keeping the gravitational field classical.

It may be very well that a quantum theory of gravitation needs new concepts and ideas. Of course, this future theory must have the today's General Relativity and Quantum Mechanics (and QFT) as limiting cases. In some sense, what everybody is doing in this domain (including string theories approach) may be something analogous to the developpment of the old quantum theory in the 10's of this century. Namely, people at that time imposed quantization conditions (the Bohr-Sommerfeld conditions) to hamiltonian mechanics but keeping the concepts of classical mechanics.

The main drawback to develop a quantum theory of gravitation is clearly the **total lack of experimental guides** for the theoretical developpment. Just from dimensional reasons, physical effects combining gravitation and quantum mechanics are relevant only at energies of the order of $M_{Planck} = \sqrt{\hbar c/G} = 1.22\,10^{16}$Tev. Such energies were available in the Universe at times $t < t_{Planck} = 5.4\,10^{-44}$sec. Anyway, as a question of principle, the construction of a quantum theory of gravitation is a problem of fundamental relevance for theoretical physics . In addition, one cannot rule out completely the possibility of some "low energy" ($E \ll M_{Planck}$) physical effect that could be experimentally tested. One may speculate about effects analogous to the presence of magnetic monopoles in some grand unified theories. [Monopoles can be detected by low energy experiments in spite of their large mass].

Let us now see what are the consequences of Heisenberg's principle in quantum mechanics combined with the notion of gravitational (Schwarzschild) radius in General Relativity. Assume we make two measurements at a very small distance Δx . Then,

$$\Delta p \sim \Delta E \sim 1/\Delta x$$

where we set $\hbar = c = 1$. For sufficiently large ΔE, particles with masses $m \sim 1/\Delta x$ will be produced. The gravitational radius of such particles are of the order

$$R_G \sim Gm \sim \frac{(l_{Planck})^2}{\Delta x}$$

where $l_{Planck} \sim 10^{-33}$ cm. Now, General Relativity allows measures at a distance Δx , provided

$$\Delta x > R_G \quad \to \quad \Delta x > \frac{(l_{Planck})^2}{\Delta x} \qquad .$$

That is,

$$\Delta x > l_{Planck} \quad \text{and} \quad m < M_{Planck} \tag{1.1}$$

This means that no measurements can be made at distances smaller than the Planck length and that no particle can be heavier than M_{Planck} . This is a simple consequence of relativistic quantum mechanics combined with General Relativity. In addition, the notion of locality and hence of spacetime becomes meaningless at the Planck scale. Notice that the equality in eq.(1.1) means that the Compton length equals the Schwarzschild radius of a particle. Since M_{Planck} is the heaviest possible particle scale, a theory valid there (necessarily involving quantum gravitation) will also be valid at any lower energy scale. One may ignore higher energy phenomena in a low energy theory, but not the opposite. In other words, a theory of quantum gravity will be a 'theory of everything'. We think that this is the **key point** on the quantization of gravity. A theory that holds till the Planck scale must describe **all** what happens at lower energies including all known particle physics as well as what we do not know yet (that is, beyond the standard model) [1]. Notice that this conclusion is totally independent of the use or not of string models. A direct important consequence of this conclusion, is that it may not make physical sense to quantize **pure gravity**. A physically sensible quantum theory cannot contain only gravitons. To give an example, a theoretical prediction for graviton-graviton scattering at energies of the order of M_{Planck} must include all particles produced in a real experiment. That is, in practice, **all** existing particles in nature, since gravity couples to all matter.

In conclusion : a consistent quantum theory of gravitation must be a finite theory [1] and must include all other interactions. That is, it must be a theory of everything (TOE). This is a very ambitious project. In particular it needs the understanding of the present desert between 1 and 10^{16} TeV. There is an additional dimensional argument about the inference Quantum Theory of Gravitation $\rightarrow$ TOE. There are only three dimensional physical magnitudes in nature: length, energy and time and correspondingly only three dimensional constants in nature: $c, \hbar$ and G. All other physical constants like $\alpha = 1/137,04...$, $M_{proton}/m_{electron}$, $\theta_{WS}, \ldots$ etc. are pure numbers and they must be calculable in a TOE. This is a formidable, but extremely appealing problem. From the theoretical side, the **only serious candidate** for a TOE is at present string theory. This is why we think that strings desserve a special attention in order to quantize gravity.

String theory is therefore an appropriate arena to work out the quantization of gravity consistently. It provides an unified theory of all interactions overcoming at the same time the nonrenormalizable character of quantum fields theories of gravity.

As a first step in the understanding of quantum gravitational phenomena in a string framework, we started in 1987 a programme of string quantization on curved spacetimes [2,9]. The investigation of strings in curved spacetimes is currently the best framework to study the physics of gravitation in the context of string theory, in spite of its limitations. First, the use of a continuous Riemanian manifold to describe the spacetime cannot be valid at scales of the order of l_{Planck}. More important, gravitational backgrounds effectively provide classical or semiclasical descriptions even if the matter backreaction to the geometry is included through semiclassical Einstein equations (or stringy corrected Einstein equations) by inserting the expectation value of the string energy-momentum tensor in the r.h.s. One would want a full quantum treatment for matter and geometry. However, to find a formulation of string theory going beyond the use of classical backgrounds is a very difficult (but fundamental) problem. One would like to derive the spacetime geometry as a classical and low energy ($\ll M_{Planck}$) limit from the solution of (quantum) string theory.

After a short introduction on strings in Minkowski and curved spacetimes, we focus on

strings in cosmological spacetimes.

Substantial results were achieved in this field since 1992. The classical behaviour of strings in FRW and inflationary spacetimes is now understood in a large extent [3]. This understanding followed the finding of various types of exact, asymptotic and numerical string solutions in FRW and inflationary spacetimes [5] - [12]. For inflationary spacetimes, the exact integrability of the string propagation equations plus the string constraints in de Sitter spacetime [4] is indeed an important help. This allowed to systematically find **exact** string solutions by soliton methods using the linear system associated to the problem (the so-called dressing method in soliton theory) and the multistring solutions [5] - [8].

In summary, three different types of behaviour are exhibited by the string solutions in cosmological spacetimes: **unstable, dual** to unstable and **stable**. For the unstable strings, the energy and size grow for large scale factors $R \to \infty$, proportional to R. For the dual to unstable strings, the energy and size blow up for $R \to 0$ as $1/R$. For stable strings, the energy and proper size are bounded. (In Minkowski spacetime, all string solutions are of the stable type). The equation of state for these string behaviours take the form

- (i) **unstable** for $R \to \infty$ $p_u = -E_u/(D-1) < 0$
- (ii) **dual to unstable** for $R \to 0, \ p_d = E_d/(D-1) > 0$.
- (iii) **stable** for $R \to \infty, \ p_s = 0$.

Here E_u and E_d stand for the corresponding string energies and $D-1$ for the number of spatial dimensions where the string solutions lives. For example, $d-1 = 1$ for a straight string, $d-1 = 2$ for a ring string, etc.

As we see above, the dual to unstable string behavior leads to the same equation of state than radiation (massless particles or hot matter). The stable string behavior leads to the equation of state of massive particles (dust or cold matter). The unstable string behavior is a purely 'stringy' phenomenon. The fact that in entails a negative pressure is however physically acceptable. For a gas of strings, the unstable string behaviour dominates in inflationary universes when $R \to \infty$ and the dual to unstable string behavior dominates for $R \to 0$.

The unstable strings correspond to the critical case of the so called *coasting universe* [19,32]. That is, classical strings provide a *concrete* realization of such cosmological models. The 'unstable' behaviour is called 'string stretching' in the cosmic string literature [20,21].

It must be stressed that while time evolves, a **given** string solution may exhibit two and even three of the above regimes one after the other (see sec. III). Intermediate behaviours are also observed in ring solutions [7,12]. That is,

$$P = (\gamma - 1)\, E \quad \text{with} \ -\frac{1}{D-1} < \gamma - 1 < +\frac{1}{D-1} \ .$$

We also report here on the exact integrability of the string equations plus the constraints in de Sitter spacetime which allows to systematically find **exact** string solutions by soliton methods and the multistring solutions. **Multistring solutions** are a new feature in curved spacetimes. That is, a single world-sheet simultaneously describes many different and independent strings. This phenomenon has no analogue in flat spacetime and appears as a consequence of the coupling of the strings with the spacetime geometry.

The world-sheet time τ turns out to be an multi-valued function of the target string time X^0 (which can be the cosmic time T, the conformal time η or for de Sitter universes it can be the hyperboloid time q^0). Each branch of τ as a function of X^0 corresponds to a different string. In flat spacetime, multiple string solutions are necessarily described by multiple world-sheets. Here, a single world-sheet describes one string, several strings or even an infinite number of different and independent strings as a consequence of the coupling with the spacetime geometry. These strings do not interact among themselves; all the interaction is with the curved spacetime. One can decide to study separately each of them (they are all different) or consider all the infinite strings together.

Of course, from our multistring solution, one *could* just choose only one interval in τ (or a subset of intervals in τ) and describe just one string (or several). This will be just a **truncation** of the solution.

The really remarkably fact is that all these infinitely many strings come **naturally together** when solving the string equations in de Sitter spacetime as we did in refs. [5] - [7].

Here, interaction among the strings (like splitting and merging) is neglected, the only interaction is with the curved background.

The multistring property appears associated to the presence of a cosmological constant (whatever be its sign) [14]. Multistring solutions have not been found in black-hole backgrounds (without cosmological constant). More recently, new classes of dynamical and stationary multistring solutions in curved spacetimes have been found and classified and their physical properties analyzed [14]. Multistrings has been found for all inflationary spacetimes [15] but not in FRW universes.

The study of string propagation in curved spacetimes provide essential clues about the physics in this context but is clearly not the end of the story. The next step beyond the investigation of **test** strings, consist in finding **self-consistently** the geometry from the strings as matter sources for the Einstein equations or better the string effective equations (beta functions). This goal is achieved in ref. [3] for cosmological spacetimes at the classical level. Namely, we used the energy-momentum tensor for a gas of strings as source for the Einstein equations and we solved them self-consistently.

To write the string equation of state we used the behaviour of the string solutions in cosmological spacetimes. Strings continuously evolve from one type of behaviour to another, as is explicitly shown by our solutions [4] - [7]. For intermediate values of R, the equation of state for gas of free strings is clearly complicated but a formula of the type:

$$\rho = \left(u_R \, R + \frac{d}{R} + s\right) \frac{1}{R^{D-1}} \tag{1.2}$$

where

$$\lim_{R\to\infty} u_R = \begin{cases} 0 & \text{FRW} \\ u_\infty \neq 0 & \text{Inflationary} \end{cases} \tag{1.3}$$

This equation of state is qualitatively correct for all R and becomes exact for $R \to 0$ and $R \to \infty$. The parameters u_R, d and s are positive constants and the u_R varies smoothly with R.

The pressure associated to the energy density (1.2) takes then the form

$$p = \frac{1}{D-1}\left(\frac{d}{R} - u_R\, R\right)\frac{1}{R^{D-1}} \tag{1.4}$$

Inserting this source into the Einstein-Friedman equations leads to a self-consistent solution for string dominated universes (see sec. VI) [3]. This solution exhibits the realistic matter dominated behaviour $R \simeq T^{2/(D-1)}$ for large times and the radiation dominated behaviour $R \simeq T^{2/D}$ for early times.

For the sake of completeness we analyze in sec. IV the effective string equations [3]. These equations have been extensively treated in the literature [28] and they are not our central aim.

It must be noticed that there is no satisfactory derivation of inflation in the context of the effective string equations. This does not mean that string theory is not compatible with inflation, but that the effective string action approach *is not enough* to describe inflation. The effective string equations are a low energy field theory approximation to string theory containing only the *massless* string modes. The vacuum energy scales to start inflation are typically of the order of the Planck mass where the effective string action approximation breaks down. One must also consider the *massive* string modes (which are absent from the effective string action) in order to properly get the cosmological condensate yielding inflation. De Sitter inflation does not emerge as a solution of the the effective string equations.

In conclusion, the effective string action (whatever be the dilaton, its potential and the central charge term) is not the appropriate framework in which to address the question of string driven inflation.

Early cosmology (at the Planck time) is probably the best place to test string theory. In one hand the quantum treatment of gravity is unavoidable at such scales and in the other hand, observable cosmological consequences are derivable from the inflationary stage. The natural gravitational background is an inflationary universe as, for instance, de Sitter spacetime. Such geometries are not string vacua. This means that conformal and Weyl symmetries are broken at the quantum level. In order to quantize consistently strings in such case, one must enlarge the physical phase space including, in particular, the factor $\exp\phi(\sigma,\tau)$ in the world-sheet metric [see eq.(2.6)]. This is a very interesting and open problem. Physically, the origin of such difficulties in quantum string cosmology comes from the fact that one is not dealing with an **empty** universe since a cosmological spacetime necessarily contains matter. In the other hand, conformal field theory techniques are till now only adapted to backgrounds for which the beta functions are identically zero, i. e. sourceless geometries. A (quantum) string theory treatment of early cosmology necessarily implies **excited** states, not just string vacua. This problem is completely open today.

The outline of these lectures is as follows. Section II presents an introduction to strings in curved spacetimes including basic notions on classical and quantum strings in Minkowski spacetime and introducing the main physical string magnitudes: energy-momentum and invariant string size.

Section III deals with the several methods of resolution of the string propagation in curved spacetimes. (In sections III.A and III.B we treat the perturbative approaches, the τ-expansion and the global solutions.

Section V deals with strings in cosmological spacetimes, the τ-expansion at work there and we present the perfect gas of strings as a model for string matter.

In section V we treat self-consistent string cosmology including the string equations of state. (Section V.A deals with general relativity, V.B with the string thermodynamics).

Section VI discuss the effective (beta functions) string equations in the cosmological perspective and the search of inflationary solutions.

In sec. VII, we briefly review the systematic construction of string solutions in de Sitter universe *via* soliton methods and the new feature of multistring solutions.

Section VIII contains the string dynamics next and inside Schwarzschild black-holes, the string behaviour near the $r = 0$ singularity and their physical properties.

II. STRINGS IN CURVED AND MINKOWSKI SPACETIMES.

Let us consider bosonic strings (open or closed) propagating in a curved D-dimensional spacetime defined by a metric $G_{AB}(X), 0 \leq A, B \leq D-1$. The action can be written as

$$S = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{g}\, g_{\alpha\beta}(\sigma,\tau)\; G_{AB}(X)\, \partial^\alpha X^A(\sigma,\tau)\, \partial^\beta X^B(\sigma,\tau) \tag{2.1}$$

Here $g_{\alpha\beta}(\sigma,\tau)$ ($0 \leq \alpha, \beta \leq 1$) is the metric in the worldsheet, α' stands for the string tension. As in flat spacetime, $\alpha' \simeq (M_{Planck})^{-2} \simeq (l_{Planck})^2$ fixes the scale in the theory. There are no other free parameters like coupling constants in string theory.

We will start considering given gravitational backgrounds $G_{AB}(X)$. That is, we start to investigate *test* strings propagating on a given spacetime. In section IV, the back reaction problem will be studied. That is, how the strings may act as source of the geometry.

String propagation in massless backgrounds other than gravitational (dilaton, antisymmetric tensor) can be investigated analogously.

The string action (2.1) classically enjoys Weyl invariance on the world sheet

$$g_{\alpha\beta}(\sigma,\tau) \rightarrow \lambda(\sigma,\tau)\, g_{\alpha\beta}(\sigma,\tau) \tag{2.2}$$

plus the reparametrization invariance

$$\sigma \rightarrow \sigma' = f(\sigma,\tau) \qquad , \qquad \tau \rightarrow \tau' = g(\sigma,\tau) \tag{2.3}$$

Here $\lambda(\sigma,\tau), f(\sigma,\tau)$ and $g(\sigma,\tau)$ are arbitrary functions.

The dynamical variables being here the string coordinates $X_A(\sigma,\tau)$, $(0 \leq A \leq D-1)$ and the world-sheet metric $g_{\alpha\beta}(\sigma,\tau)$.

Extremizing S with respect to them yields the classical equations of motion:

$$\begin{aligned} \partial^\alpha[\sqrt{g}\, G_{AB}(X)\; \partial_\alpha X^B(\sigma,\tau)] = {} & \frac{1}{2}\sqrt{g}\; \partial_A G_{CD}(X)\, \partial_\alpha X^C(\sigma,\tau)\, \partial^\alpha X^D(\sigma,\tau) \\ & 0 \leq A \leq D-1 \end{aligned} \tag{2.4}$$

$$\begin{aligned} T_{\alpha\beta} \;\equiv\; & G_{AB}(X)[\, \partial_\alpha X^A(\sigma,\tau)\, \partial_\beta X^B(\sigma,\tau) \\ & - \frac{1}{2} g_{\alpha\beta}(\sigma,\tau)\, \partial_\gamma X^A(\sigma,\tau)\, \partial^\gamma X^B(\sigma,\tau)\,] = 0 \quad , \quad 0 \leq \alpha, \beta \leq 1. \end{aligned} \tag{2.5}$$

Eqs. (2.5) contain only first derivatives and are therefore a set of constraints. Classically, we can always use the reparametrization freedom (2.3) to recast the world-sheet metric on diagonal form

$$g_{\alpha\beta}(\sigma,\tau) = \exp[\phi(\sigma,\tau)] \; \mathrm{diag}(-1,+1) \tag{2.6}$$

In this conformal gauge, eqs. (2.4) - (2.5) take the simpler form:

$$\partial_{-+}X^A(\sigma,\tau) + \Gamma^A_{BC}(X)\,\partial_+X^B(\sigma,\tau)\,\partial_-X^C(\sigma,\tau) = 0\;, \quad 0 \leq A \leq D-1, \tag{2.7}$$

$$T_{\pm\pm} \equiv G_{AB}(X)\,\partial_\pm X^A(\sigma,\tau)\,\partial_\pm X^B(\sigma,\tau) \equiv 0\,, \quad T_{+-} \equiv T_{-+} \equiv 0 \tag{2.8}$$

where we introduce light-cone variables $x_\pm \equiv \sigma \pm \tau$ on the world-sheet and where $\Gamma^A_{BC}(X)$ stand for the connections (Christoffel symbols) associated to the metric $G_{AB}(X)$.

Notice that these equations in the conformal gauge are still invariant under the conformal reparametrizations:

$$\sigma + \tau \to \sigma' + \tau' = f(\sigma+\tau) \qquad , \qquad \sigma - \tau \to \sigma' - \tau' = g(\sigma-\tau) \tag{2.9}$$

Here $f(x)$ and $g(x)$ are arbitrary functions.

The string boundary conditions in curved spacetimes are identical to those in Minkowski spacetime. That is,

$$\begin{aligned} X^A(\sigma+2\pi,\tau) &= X^A(\sigma,\tau) \quad \text{closed strings} \\ \partial_\sigma X^A(0,\tau) &= \partial_\sigma X^A(\pi,\tau) = 0 \quad \text{open strings.} \end{aligned} \tag{2.10}$$

A. A brief review on strings in Minkowski spacetime

In flat spacetime eqs.(2.7) become linear

$$\partial_{-+}X^A(\sigma,\tau) = 0\;, \quad 0 \leq A \leq D-1, \tag{2.11}$$

and one can solve them explicitly as well as the quadratic constraint (2.8) [see below]:

$$\left[\partial_\pm X^0(\sigma,\tau)\right]^2 - \sum_{j=1}^{D-1}\left[\partial_\pm X^j(\sigma,\tau)\right]^2 = 0 \tag{2.12}$$

The solution of eqs.(2.11) is usually written for closed strings as

$$X^A(\sigma,\tau) = q^A + 2p^A\alpha'\tau + i\sqrt{\alpha'}\sum_{n\neq 0}\frac{1}{n}\{\alpha^A_n \exp[in(\sigma-\tau)] + \tilde{\alpha}^A_n \exp[-in(\sigma+\tau)]\} \tag{2.13}$$

where q^A and p^A stand for the string center of mass position and momentum and α^A_n and $\tilde{\alpha}^A_n$ describe the right and left oscillator modes of the string, respectively. Since the string coordinates are real,

$$\bar{\alpha}^A_n = \alpha^A_{-n} \quad , \quad \bar{\tilde{\alpha}}^A_n = \tilde{\alpha}^A_{-n}$$

This resolution is no more possible in general for curved spacetime where the equations of motion (2.7) are non-linear. In that case, right and left movers interact with themselves and with each other.

In Minkowski spacetime we can also write the solution of the string equations of motion (2.11) in the form

$$X^A(\sigma,\tau) = l^A(\sigma+\tau) + r^A(\sigma-\tau) \tag{2.14}$$

where $l^A(x)$ and $r^A(x)$ are arbitrary functions. Now, making an appropriate conformal transformation (2.9) we can turn any of the string coordinates $X^A(\sigma,\tau)$ (but only one of them) into a constant times τ. The most convenient choice is the light-cone gauge where

$$U \equiv X^0 - X^1 = 2\,p^U \alpha' \tau. \tag{2.15}$$

That is, there are no string oscillations along the U direction in the light-cone gauge. We have still to impose the constraints (2.12). In this gauge they take the form

$$\pm\, 2\alpha' p^U\, \partial_\pm V(\sigma,\tau) = \sum_{j=2}^{D-1} \left[\partial_\pm X^j(\sigma,\tau)\right]^2 \tag{2.16}$$

where $V \equiv X^0 + X^1$. This shows that V is not an independent dynamical variable since it expresses in terms of the transverse coordinates $X^2, \ldots, X^{D-1}$. Only q^V is an independent quantity.

The physical picture of a string propagating in Minkowski spacetime clearly emerges in the light-cone gauge. The gauge condition (2.15) tells us that the string 'time' τ is just proportional to the physical null time U. Eqs.(2.13) shows that the string moves as a whole with constant speed while it oscillates around its center of mass. The oscillation frequencies are all integers multiples of the basic one. The string thus possess an infinite number of normal modes; $\alpha_n^A, \tilde{\alpha}_n^A$ classically describe their oscillation amplitudes. Only the modes in the direction of the transverse coordinates $X^2, \ldots, X^{D-1}$ are physical. This is intuitively right, since a longitudinal or a temporal oscillation of a string is meaningless. In summary, the string in Minkowski spacetime behaves as an extended and composite relativistic object formed by a $2(D-2)$-infinite set of harmonic oscillators.

Integrating eq.(2.16) on σ from 0 to 2π and inserting eq. (2.13) yields the classical string mass formula:

$$m^2 \equiv p^U p^V - \sum_{j=2}^{D-1} (p^j)^2 = \frac{1}{\alpha'} \sum_{j=2}^{D-1} \sum_{n=1}^{\infty} \left[\alpha_n^j\, \alpha_{-n}^j + \tilde{\alpha}_n^j\, \tilde{\alpha}_{-n}^j\right] \tag{2.17}$$

We explicitly see how the mass of a string depends on its excitation state. The classical string spectrum is continuous as we read from eq.(2.17). It starts at $m^2 = 0$ for an unexcited string: $\alpha_n^j = \tilde{\alpha}_n^j = 0$ for all n and j.

The independent string variables are:
the transverse amplitudes $\{\alpha_n^j, \tilde{\alpha}_n^j, n\varepsilon\mathcal{Z}, n \neq 0,\ 2 \leq j \leq D-1\}$,
the transverse center of mass variables $\{q^j, p^j,\ 2 \leq j \leq D-1\}$,
q^V and p^U.

Up to now we have considered a classical string.
At the quantum level one imposes the canonical commutation relations (CCR)

$$[\alpha^i_n, \alpha^j_m] = n\ \delta_{n,-m}\, \delta^{i,j} \quad , \quad [\tilde{\alpha}^i_n, \alpha^j_m] = n\ \delta_{n,-m}\, \delta^{i,j} \ ,$$

$$[\tilde{\alpha}^i_n, \alpha^j_m] = 0 \ ,$$

$$[q^i, p^j] = i\ \delta^{i,j} \quad , \quad [q^V, p^U] = i \tag{2.18}$$

All other commutators being zero. An order prescription is needed to unambiguously express the different physical operators in terms of those obeying the CCR. The symmetric ordering is the simplest and more convenient.

The space of string physical states is the the tensor product of the Hilbert space of the $D-1$ center of mass variables $q_V, p_U, \{q^j, p^j, 2 \leq j \leq D-1\}$, times the Fock space of the harmonic transverse modes. The string wave function is then the product of a center of mass part times a harmonic oscillator part. The center of mass can be taken, for example, as a plane wave. The harmonic oscillator part can be written as the creation operators ${\alpha^j_n}^\dagger, {\tilde{\alpha}^j_n}^\dagger,\ n \geq 1,\ 2 \leq j \leq D-1$ acting on the oscillator ground state $|0>$. This state is defined as usual by

$$\alpha^j_n\ |0>= \tilde{\alpha}^j_n\ |0>= 0, \quad \text{for all } n \geq 1,\ 2 \leq j \leq D-1$$

Notice that a string describes **one particle**. The kind of particle described depends on the oscillator wave function. The mass and spin can take an infinite number of different values. That is, there is an infinite number of different possibilities for the particle described by a string.

Let us consider the quantum mass spectrum. Upon symmetric ordering the mass operator becomes,

$$m^2 = \frac{1}{2\alpha'} \sum_{j=2}^{D-1} \sum_{n=1}^{\infty} \left[\alpha^j_n\, \alpha^j_{-n} + \alpha^j_{-n}\, \alpha^j_n + \tilde{\alpha}^j_n\, \tilde{\alpha}^j_{-n} + \tilde{\alpha}^j_{-n}\, \tilde{\alpha}^j_n \right] \ . \tag{2.19}$$

Using the commutation rules (2.18) yields

$$m^2 = \frac{D-2}{\alpha'} \sum_{n=1}^{\infty} n + \frac{1}{2\alpha'} \sum_{j=2}^{D-1} \sum_{n=1}^{\infty} \left[{\alpha^j_n}^\dagger\, \alpha^j_n + {\tilde{\alpha}^j_n}^\dagger\, \tilde{\alpha}^j_n \right] \tag{2.20}$$

The divergent sum in the first term can be defined through analytic continuation of the zeta function

$$\zeta(z) \equiv \sum_{n=1}^{\infty} \frac{1}{n^z} \tag{2.21}$$

One finds $\zeta(-1) = -1/12$ [30]. Thus,

$$m^2 = -\frac{D-2}{12\alpha'} + \frac{1}{2\alpha'} \sum_{j=2}^{D-1} \sum_{n=1}^{\infty} \left[{\alpha^j_n}^\dagger\, \alpha^j_n + {\tilde{\alpha}^j_n}^\dagger\, \tilde{\alpha}^j_n \right] \tag{2.22}$$

Hence, the string ground state $|0>$ has a negative mass squared

$$m^2_0 = -\frac{D-2}{12\alpha'} \tag{2.23}$$

Such particles are called tachyons and exhibit unphysical behaviours. When fermionic degrees of freedom are associated to the string the ground state becomes massless (superstrings) [29].

Notice that the appearance of a negative mass square yields a dispersion relation $E^2 = p^2 - |m_0^2|$ similar to classical waves when gravity (even newtonian) is taken into account (Jeans unstabilities) [31].

Let us consider now excited states.

The constraints (2.12) integrated on σ from 0 to 2π impose

$$\sum_{j=2}^{D-1} \sum_{n=1}^{\infty} (\alpha_n^j)^\dagger \, \alpha_n^j = \sum_{j=2}^{D-1} \sum_{n=1}^{\infty} \left(\tilde{\alpha_n^j}\right)^\dagger \, \tilde{\alpha}_n^j \tag{2.24}$$

on the physical states. This means that the number of left and right modes coincide in all physical states.

The first excited state is then described by

$$|i, j >= \left(\tilde{\alpha_1^i}\right)^\dagger (\alpha_1^j)^\dagger \; |0 > \tag{2.25}$$

times the center of mass wave function. We see that this wavefunction is a symmetric tensor in the space indices i, j. It describes therefore a spin two particle plus a spin zero particle (the trace part).

From eqs.(2.22-2.25) follows that

$$m^2 \; |i, j >= -\frac{D - 26}{12\alpha'} |i, j > \tag{2.26}$$

This state is then a massless particle only for $D = 26$. In such critical dimension we have then a graviton (massless spin 2 particle) and a dilaton (massless spin 0 particle) as string modes of excitation. For superstrings the critical dimension turns to be $D = 10$ [29].

We shall consider, as usual, that only four space-time dimensions are uncompactified. That is, we shall consider the strings as living on the tensor product of a curved four dimensional space-time with lorentzian signature and a compact space which is there to cancel the anomalies. From now on strings will propagate in the curved (physical) four dimensional space-time. However, we will find instructive to study the case where this curved space-time has dimensionality D, where D may be 2, 3, 4 or arbitrary.

B. The string energy-momentum tensor and the string invariant size

The spacetime string energy-momentum tensor follows (as usual) by taking the functional derivative of the action (2.1) with respect to the metric G_{AB} at the spacetime point X. This yields,

$$\sqrt{-G} \; T^{AB}(X) = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \left(\dot{X}^A \dot{X}^B - X'^A X'^B\right) \delta^{(D)}(X - X(\sigma, \tau)) \tag{2.27}$$

where dot and prime stands for $\partial/\partial\tau$ and $\partial/\partial\sigma$, respectively.

Notice that X in eq.(2.27) is just a spacetime point whereas $X(\sigma, \tau)$ stands for the string dynamical variables. One sees from the Dirac delta in eq.(2.27) that $T^{AB}(X)$ vanishes unless

X is exactly on the string world-sheet. We shall not be interested in the detailed structure of the classical strings. It is the more useful to integrate the energy-momentum tensor (2.27) on a volume that completely encloses the string. It takes then the form [17]

$$\Theta^{AB}(X^0) = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \left(\dot{X}^A \dot{X}^B - X'^A X'^B \right) \delta(X^0 - X^0(\tau, \sigma)). \tag{2.28}$$

When X^0 depends only on τ, we can easily integrate over τ with the result,

$$\Theta^{AB}(X^0) = \frac{1}{2\pi\alpha' |\dot{X}^0(\tau)|} \int_0^{2\pi} d\sigma \left[\dot{X}^A \dot{X}^B - X'^A X'^B \right]_{\tau=\tau(X^0)} \tag{2.29}$$

Another relevant physical magnitude for strings is the invariant size. We define the invariant string size ds^2 using the metric induced on the string world-sheet:

$$ds^2 = G_{AB}(X)\, dX^A\, dX^B \tag{2.30}$$

Inserting $dX^A = \partial_+ X^A\ dx^+ + \partial_- X^A\ dx^-$, into eq.(2.30) and taking into account the constraints (2.8) yields

$$ds^2 = 2\, G_{AB}(X)\, \partial_+ X^A\, \partial_- X^B \left(d\tau^2 - d\sigma^2 \right) = G_{AB}(X)\, \dot{X}^A\, \dot{X}^B \left(d\tau^2 - d\sigma^2 \right). \tag{2.31}$$

Thus, we define the string size as the integral of

$$S(\sigma, \tau) \equiv \sqrt{G_{AB}(X)\, \dot{X}^A\, \dot{X}^B} \tag{2.32}$$

over σ at fixed τ. For a causal choice of the string time τ we must have

$$G_{AB}(X)\, \dot{X}^A\, \dot{X}^B \geq 0 \tag{2.33}$$

The equality sign here corresponds to a string behaving as radiation. Such type of solutions always exist. For example any σ-independent solution of eqs.(2.7-2.8). Such solutions describe massless geodesics.

Notice that the trace of the energy-momentum tensor eq.(2.27) is just the integral of $S(\sigma, \tau)^2$,

$$\begin{aligned} \sqrt{-G}\ T^A_A(X) &= \frac{1}{\pi\alpha'} \int d\sigma d\tau\ G_{AB}(X) \dot{X}^A \dot{X}^B\ \delta^{(D)}(X - X(\sigma, \tau)) \\ &= \frac{1}{\pi\alpha'} \int d\sigma d\tau\ S(\sigma, \tau)^2\ \delta^{(D)}(X - X(\sigma, \tau)) \end{aligned} \tag{2.34}$$

For strings behaving as radiation both $T^A_A(X)$ and S vanish.

C. Simple String Solutions in Minkowski Spacetime

Let us now consider a circular string as a simple example of a string solution in Minkowski spacetime.

$$X^0(\sigma, \tau) = \alpha' E \tau \quad , \quad X^3(\sigma, \tau) = \alpha' p \tau$$

$$X^1(\sigma,\tau) = \alpha' m \cos\tau \cos\sigma = \frac{\alpha' m}{2}\left[\cos(\tau+\sigma)+\cos(\tau-\sigma)\right] \tag{2.35}$$

$$X^2(\sigma,\tau) = \alpha' m \cos\tau \sin\sigma = \frac{\alpha' m}{2}\left[\sin(\tau+\sigma)-\sin(\tau-\sigma)\right]$$

This is obviously a solution of eqs.(2.11) where only the modes $n = \pm 1, j = 1,2$ are excited. The constraints (2.12) yields

$$E^2 = p^2 + m^2$$

Eqs.(2.35) describe a circular string in the X^1, X^2 plane, centered in the origin and with an oscillating radius $\rho(\tau) = \alpha' m \cos\tau$. In addition the string moves uniformly in the z-direction with speed p/E. (That is, p is its momentum in the z-direction). The oscillation amplitude m can be identified with the string mass and E with the string energy. Notice that the string time τ is here proportional to the physical time X^0 [this solution is not in the light-cone gauge (2.15)].

It is instructive to compute the integrated energy-momentum tensor (2.29) for this string solution. We find in the rest frame ($p = 0$) that it takes the fluid form

$$\Theta_A^B = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.36}$$

where

$$\rho = E = m \quad , \quad p = -\frac{m}{2}\cos(2\tau) \tag{2.37}$$

We see that the total energy coincides with m as one could expect and that the (space averaged) pressure oscillates around zero. That is, the string pressure goes through positive and negative values. The time average of p on a period vanishes. The string behaves then as cold matter (massive particles).

The upper value of p equals $E/2$. This is precisely the relation between E and p for radiation (massless particles). (Notice that this circular strings lives on a two-dimensional plane). The lower value of p correspond to the limiting value allowed by the strong energy condition in General Relativity [16]. We shall see below that these two extreme values of p appear for strings in general cosmological spacetimes.

The invariant size of the string solution (2.35) follows by inserting eq.(2.35) into eq.(2.31). We find

$$ds^2 = (\alpha' m)^2 \left(d\tau^2 - d\sigma^2\right) \tag{2.38}$$

Therefore, this string solution has a constant size $2\pi\alpha' m$.

Another simple but instructive solution in Minkowski spacetime is a rotating straight string (a rotating rod) given by

$$X^0(\sigma,\tau) = \alpha' m \tau \quad ,$$

$$X^1(\sigma,\tau) = \alpha' m \cos\tau \cos\sigma \ , \tag{2.39}$$

$$X^2(\sigma,\tau) = \alpha' m \sin\tau \cos\sigma \ ,$$

or in polar coordinates

$$\rho = \alpha' m \mid \cos\sigma \mid \quad , \quad \phi = \tau \ .$$

That is, a straight string on the $X^1 - X^2$ plane rotating around the origin with an angular speed $\frac{1}{\alpha' m}$. Eqs. (2.39) identically fulfil the string equations and constraints (2.11-2.12).

The energy momentum tensor for this rotating string takes the form:

$$\Theta_A^B = \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & \frac{m}{2}\cos(2\tau) & \frac{m}{2}\sin(2\tau) & 0 \\ 0 & \frac{m}{2}\sin(2\tau) & -\frac{m}{2}\cos(2\tau) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{2.40}$$

This means an energy density $\rho = m$. The spatial part of Θ_A^B becomes diagonal in a rotating frame and has $\pm\frac{m}{2}$ as eigenvalues. That is, we again find $\pm\frac{\rho}{2}$ as extreme values for the pressure in two space dimensional string solutions. The time average of the stress tensor Θ_i^j here vanishes indicating zero pressure (cold matter behaviour).

One obtains the invariant size for a generic solution in Minkowski spacetime inserting the general solution (2.13) into $\partial_+ X^A \partial_- X_A$. This gives constant plus oscillatory terms. In any case the invariant string size is always **bounded** in Minkowski spacetimes. We shall see how differently behave strings in curved spacetimes.

III. HOW TO SOLVE THE STRING EQUATIONS OF MOTION IN CURVED SPACETIMES?

There is no general method to solve the string equations of motion and constraints for arbitrary curved spacetime.

The so-called τ-expansion method provides **exact local** solutions for any background. The basic idea goes as follows. Suppose one is interested on the string behaviour near a given point of the curved spacetime. Then, one chooses a conformal gauge such that $\tau = 0$ corresponds to such a point. For example, to study strings in cosmological spacetimes near the initial singularity (cosmic time $T = 0$), one chooses $T(\sigma,\tau)$ such that $T(\sigma,0) = 0$. It is shown below that this is indeed possible for generic string solutions. This expansion was developped first in ref. [10,11] for inflationary universes.

Similarly, in order to study strings near the black hole singularity $r = 0$, one chooses $r(\sigma,\tau)$ such that $r(\sigma,0) = 0$.

Once this gauge choice is done, the string equations of motion and constraints can be solved in powers of τ. These powers may not be integer powers. For example, one finds powers of $\tau^{-\frac{2}{\alpha+1}}$ in FRW universes with scale factor $R(T) = c\, T^\alpha$. For Schwarzschild black holes powers of $\tau^{\frac{2}{5}}$ appear.

An approximate but general method is the expansion around center of mass solutions [2,9]. In this method one starts from an exact solution of the geodesic equations

$$\ddot{q}^A(\tau) + \Gamma^A_{BC}(q)\,\dot{q}^B(\tau)\,\dot{q}^C(\tau) = 0 \tag{3.1}$$

The world-sheet time variable is here identified with the proper time of the center of mass trajectory. [Notice that eqs.(3.1) just follow from the string equations (2.7) by dropping the σ-dependence].

Then one develops in perturbations around it. That is, one sets

$$X^A(\sigma,\tau) = q^A(\tau) + \eta^A(\sigma,\tau) + \xi^A(\sigma,\tau) + \ldots \tag{3.2}$$

Here $\eta^A(\sigma,\tau)$ obeys the linearized perturbation around $q^A(\sigma,\tau)$ and $\xi^A(\sigma,\tau)$ the second order perturbation equations [2]. These fluctuations obey coupled ordinary differential equations that can be written systematically inserting eq.(3.2) into eqs.(2.7-2.8). [See ref. [9] for more details].

Another general approximation method is the null string approach [22]. In such approach the string equations of motion and constraints are systematically expanded in powers of c (the speed of light in the world-sheet). This corresponds to a small string tension expansion. At zeroth order, the string is effectively equivalent to a continuous beam of massless particles labelled by the parameter σ. The points on the string do not interact between them but they interact with the gravitational background.

For several spacetimes one can construct explicit string solutions using specific properties of the background. This is the case of singular plane waves, shock-waves, conical spacetimes and the de Sitter universe. The string equations of motion and constraints in the de Sitter spacetime are integrable in the inverse scattering sense as shown in ref. [4]. [The de Sitter universe is a symmetric space and hence the string equations there correspond to a two dimensional integrable sigma model].

A. The τ-expansion

Let us consider the intersection of the world-sheet with a singular or non-singular point (or surface) of the spacetime like $T(\sigma,\tau) = 0$ or $T(\sigma,\tau) = T_o$ in a cosmological spacetime or $r(\sigma,\tau) = 0$ or $r(\sigma,\tau) = r_o$ in a Schwarzschild black-hole.

We can write the curve describing such intersection with the world-sheet as

$$x_+ = \chi(x_-), \tag{3.3}$$

whenever this intersection is nondegenerate. (Here $x_\pm \equiv \tau \pm \sigma$).

Upon a conformal transformation,

$$x_+ \to x'_+ = f(x_+) \quad , \quad x_- \to x'_- = g(x_-), \tag{3.4}$$

we can map the curve (3.3) into $\tau' = 0$ by an appropriate choice of f and g [18]. For example, we can choose

$$f(x_+) = x_+ \quad , \quad g(x_-) = -\chi(x_-).$$

This defines our choice of gauge. From now on, we rename τ' and σ' by τ and σ, respectively. Notice that this choice does not completely fix the gauge. We can still perform transformations that leave the line $\tau = 0$ unchanged. This is the case for the following class of conformal mappings

$$x_+ \rightarrow x'_+ = \varphi(x_+) \quad , \quad x_- \rightarrow x'_- = -\varphi(-x_-), \tag{3.5}$$

where $\varphi(x)$ is an arbitrary function. Eq. (3.5) can be written as,

$$\begin{aligned}
\tau' &= \frac{1}{2}\left[\varphi(\tau+\sigma) - \varphi(\sigma-\tau)\right] = \tau\,\varphi'(\sigma) \; + \frac{1}{6}\tau^3\;\varphi'''(\sigma) + O(\tau^4) \\
\sigma' &= \frac{1}{2}\left[\varphi(\tau+\sigma) + \varphi(\sigma-\tau)\right] = \varphi(\sigma) \; + \frac{1}{2}\tau^2\;\varphi''(\sigma) + O(\tau^4)
\end{aligned} \tag{3.6}$$

The transformations (3.5) represent a diagonal subgroup of the set of left-right conformal transformations (3.4).

In summary, any (non-degenerate) intersection of the world-sheet with a spacetime submanifold $T(\sigma,\tau) = T_o$ can be mapped into $\tau = 0$. This mapping is not unique, it is invariant under the diagonal conformal transformations.

Once this gauge has been imposed one studies the string equations of motion and constraints (2.7-2.8) in powers of τ. The equations themselves determine the precise values of the powers [10,11,18].

B. Global Solutions

There is no general method to find solutions valid in the whole world-sheet. However, many global solutions have been found in physically relevant spacetimes.

First, there are spacetimes where the **general** solution of the string equations and constraints has been found. That is, shock-waves [44], singular plane waves [46] and conical spacetimes [47].

Second, by making specific ansatz according to the symmetry of the background, the string equations of motion can be reduced to ordinary differential equations. Then, these ordinary differential equations can be solved globally by numerical methods. In this way, solutions valid in the whole worldsheet has been found in cosmological spacetimes and black holes [7,12] - [14].

Third, the de Sitter spacetime can be treated by inverse scattering methods. In this way exact string solutions has been constructed systematically (see sec. VII and refs. [5] - [8]).

In all cases where global solutions can be found, the τ-expansion results are confirmed.

IV. STRING PROPAGATION IN COSMOLOGICAL SPACETIMES

We obtain in this section physical string properties from the string solutions in cosmological spacetimes.

We consider strings in spatially homogeneous and isotropic universes with metric

$$ds^2 = (dT)^2 - R(T)^2 \sum_{i=1}^{D-1} (dX^i)^2 \; , \tag{4.1}$$

where T is the cosmic time and the function $R(T)$ is called the scale factor. In terms of the conformal time

$$\eta = \int^T \frac{dT}{R(T)} \, , \tag{4.2}$$

the metric (4.1) takes the form

$$ds^2 = R(\eta)^2 \left[(d\eta)^2 - \sum_{i=1}^{D-1} (dX^i)^2 \right] \, . \tag{4.3}$$

The classical string equations of motion can be written here as

$$\begin{aligned} &\partial^2 T + \; R(T) \; \frac{\mathrm{d}R}{\mathrm{d}T} \; \sum_{i=1}^{D-1} (\partial_\mu X^i)^2 = 0 \, , \\ &\partial_\mu \left[R^2 \partial^\mu X^i \right] = 0 \, , \qquad 1 \leq i \leq D-1 \, , \end{aligned} \tag{4.4}$$

and the constraints are

$$T_{\pm\pm} = (\partial_\pm T)^2 - R(T)^2 \; (\partial_\pm X^i)^2 = 0 \, . \tag{4.5}$$

The most relevants universes correspond to power type scale factors. That is,

$$R(T) = a \; T^\alpha = A \; \eta^{k/2} \, , \tag{4.6}$$

where $\alpha = \frac{k}{k+2}$ and $k = \frac{2\alpha}{1-\alpha}$.

For different values of the exponents we have either FRW or inflationary universes.

$$\text{FRW}: \; 0 < k \leq \infty, \; 0 < \alpha \leq 1 = \begin{cases} \alpha = 1/2, \; k = 2, & \text{radiation dominated,} \\ \alpha = 2/3, \; k = 4, & \text{matter dominated,} \\ \alpha = 1, \; k = \infty, & \text{`stringy'.} \end{cases}$$

$$\text{Inflationary}: -\infty < k < 0, \; \alpha < 0 \text{ and } \alpha > 1 = \begin{cases} \alpha = \infty, \; k = -2, & R(T) = e^{HT}, \text{ de Sitter,} \\ \alpha > 1, \; k < -2, & \text{power inflation,} \\ \alpha < 0, \; -2 < k < 0, & \text{superinflationary.} \end{cases}$$

The denomination 'stringy' comes from the fact that such backgrounds follow as solution of the string effective equations [24]. Inflationary universes are those with accelerated expansion. That is,

$$\frac{d^2 R(T)}{dT^2} > 0 \, .$$

The string equations of motion and constraints take then the following form using the conformal time η:

$$\begin{aligned} &\ddot{\eta} - \eta'' + \frac{k}{2\eta} \left\{ \dot{\eta}^2 - \eta'^2 + \sum_{i=1}^{D-1} \left[(\dot{X}^i)^2 - (X'^i)^2 \right] \right\} = 0 \, , \\ &\ddot{X}^i - X''^i + \frac{k}{\eta} \left[\dot{\eta} \dot{X}^i - \eta' X'^i \right] = 0 \, , \quad 1 \leq i \leq D-1 \, , \\ &(\dot{\eta} \pm \eta')^2 - \sum_{i=1}^{D-1} (\dot{X}^i \pm X'^i)^2 = 0 \, , \end{aligned} \tag{4.7}$$

where prime and dot stand for ∂_σ and ∂_τ, respectively.

As we will see below, once appropriately averaged $\Theta^{AB}(X)$ takes the fluid form for string solutions in cosmological spacetimes, allowing us to define the string pressure p and energy density ρ :

$$< \Theta_A^B >= \left(\begin{array}{cccc} \rho & 0 & \cdots & 0 \\ 0 & -p & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & -p \end{array} \right) . \tag{4.8}$$

Notice that the continuity equation

$$D^A \, \Theta_A^B = 0 \; ,$$

takes here the form

$$\frac{d\rho}{dT} + (D-1) \, H \, (p+\rho) = 0 \tag{4.9}$$

where $H \equiv \frac{1}{R}\frac{dR}{dT}$.

For an equation of state of the type of a perfect fluid, that is

$$p = (\gamma - 1) \, \rho \qquad , \qquad \gamma = \text{ constant}, \tag{4.10}$$

eqs.(4.9) and (4.10) can be easily integrated with the result

$$\rho = \rho_0 \; R^{\gamma(1-D)} \quad . \tag{4.11}$$

For $\gamma = 1$ this corresponds to cold matter ($p = 0$) and for $\gamma = \frac{D}{D-1}$ this describes radiation with $p = \frac{\rho}{D-1}$.

The speed of sound in a fluid is given by

$$v_s = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)|_s} = \sqrt{\gamma - 1}$$

This relation makes sense for $1 \geq \gamma - 1 \geq 0$. For $\gamma - 1 < 0$, there are no sound waves since the perturbations in p and ρ obey elliptic evolution equations. For $\gamma - 1 > 1$ causality would be violated. Actually, for strings we always find $-\frac{1}{D-1} \leq \gamma - 1 \leq \frac{1}{D-1}$.

A. Strings in cosmological universes: the τ-expansion at work

Let us consider strings in inflationary universes with scale factor (4.6) and $k < 0$. In order to apply the τ-expansion we fix the gauge such that

$$\eta(\tau = 0, \sigma) = 0 \; . \tag{4.12}$$

As explained above, this is always possible for generic string solutions and it leaves still the freedom of the transformations (3.5). Notice that $\eta \to 0$ corresponds in the inflationary case to large scale factors $R \to \infty$.

The behaviour of $\eta(\tau,\sigma)$ and $X^i(\tau,\sigma)$ for $\tau \to 0$ follows from eq.(4.7) where we use eq.(4.12) and assume that $X^i(\tau,\sigma)$ is regular at $\tau = 0$. One finds [11]

$$\begin{aligned}
\eta(\tau,\sigma) &\overset{\tau\to 0}{=} \eta_o(\sigma)\ \tau\ \left[1+O(\tau^2)\right] + \lambda_o(\sigma)\ \tau^{2-k}\ \left[1+O(\tau^2)\right] \\
&+\ \zeta_o(\sigma)\ \tau^{1-2k}\ \left[1+O(\tau^2)\right] + O(\tau^{1-3k})\ , \\
X^i(\tau,\sigma) &\overset{\tau\to 0}{=} A^i(\sigma)\left[1+O(\tau^2)\right] + \tau^{1-k}\ B^i(\sigma)\ \left[1+O(\tau^2,\tau^{1-k})\right]\ , \\
& 1 \le i \le D-1\ .
\end{aligned} \tag{4.13}$$

The solutions appear as a series in powers of τ^2 and τ^{1-k}. In the special case where k is rational, say $k = -\frac{l}{n}$, l, n =integers, logarithmic terms in τ appear in addition. This happens, for example in de Sitter spacetime ($k = -2$) and in Minkowski spacetime ($k = 0$).

The coefficients in eq.(4.15) result related as follows

$$\begin{aligned}
&\sum_{i=1}^{D-1} B^i(\sigma)\ A'^i(\sigma) = 0\ ,\ \eta_o(\sigma) = \sqrt{\sum_{i=1}^{D-1} \left[A'^i(\sigma)\right]^2} \\
&\lambda_o(\sigma) = \frac{2k}{(2-k)(1-k)}\ \frac{\sum_{i=1}^{D-1} B'^i(\sigma)\ A'^i(\sigma)}{\eta_o(\sigma)}\ .
\end{aligned} \tag{4.14}$$

Moreover, one can use the residual conformal invariance (3.5) to set $\eta_o(\sigma) \equiv 1$. One finally obtains for inflationary universes [11],

$$\begin{aligned}
\eta(\tau,\sigma) &\overset{\tau\to 0}{=} \tau\ \left[1+O(\tau^2)\right] + \lambda_o(\sigma)\ \tau^{2-k}\ \left[1+O(\tau^2)\right] \\
&+\ \zeta_o(\sigma)\ \tau^{1-2k}\ \left[1+O(\tau^2)\right] + O(\tau^{1-3k})\ , \\
X^i(\tau,\sigma) &\overset{\tau\to 0}{=} A^i(\sigma)\left[1+O(\tau^2)\right] + \tau^{1-k}\ B^i(\sigma)\ \left[1+O(\tau^2,\tau^{1-k})\right]\ , \\
& 1 \le i \le D-1\ .
\end{aligned} \tag{4.15}$$

where

$$\begin{aligned}
&\sum_{i=1}^{D-1} B^i(\sigma)\ A'^i(\sigma) = 0 \quad , \quad \sum_{i=1}^{D-1} \left[A'^i(\sigma)\right]^2 = 1 \\
&\lambda_o(\sigma) = \frac{2k}{(2-k)(1-k)} \sum_{i=1}^{D-1} B'^i(\sigma)\ A'^i(\sigma) \quad , \quad \zeta_o(\sigma) = \frac{(1-k)^2}{2(1-2k)} \sum_{i=1}^{D-1} \left[B'^i(\sigma)\right]^2\ .
\end{aligned} \tag{4.16}$$

Here $A^i(\sigma),\ B^i(\sigma),\ 1 \le i \le D-1$ are the initial ($\tau = 0$) string coordinates and momenta.

We see that the solution depends on $2(D-2)$ independent functions among the $A^i(\sigma)$ and $B^i(\sigma)$, $1 \le i \le D-1$. All coefficients (including the higher orders not written in eq.(4.15)) express in terms of the $A^i(\sigma)$ and $B^i(\sigma)$. Therefore, the counting of degrees of freedom turns to be the same as in Minkowski spacetime: only the $2(D-2)$ **transverse** coordinates are physical.

It must be noticed that

$$\dot{X}^i(\tau,\sigma) \overset{\tau\to 0}{=} (1-k)\ \tau^{-k}\ B^i(\sigma)\ \left[1+O(\tau^2,\tau^{1-k})\right] \to 0$$

$$X'^i(\tau,\sigma) \overset{\tau\to 0}{=} A'^i(\sigma) \neq 0$$

That is, $\partial_\sigma X^i$ is **larger** than $\partial_\tau X^i$ for $R \to \infty$. This is the opposite to a point particle behaviour. For a point particle, $\partial_\sigma X^i \equiv 0$ and $\partial_\tau X^i = p^i \neq 0$.

Let us now apply the τ-expansion to strings in FRW universes. That is $k > 0$ in the scale factor (4.6).

We fix again the gauge according to eq.(4.12). It must be noticed that now $\eta \to 0$ corresponds to $R \to 0$ since $k > 0$. That is the τ-expansion applies near the singularity (big bang) of the spacetime.

After calculations analogous to the inflationary case, one finds from eqs.(4.7) for FRW universes ($k > 0$) [11]

$$\begin{aligned} \eta(\tau,\sigma) &\overset{\tau\to 0}{=} \tau^{\frac{1}{k+1}} \left[1+O(\tau^2)\right] + \eta_1(\sigma)\,\tau^{2-\frac{1}{k+1}}\left[1+O(\tau^2)\right] \\ X^i(\tau,\sigma) &\overset{\tau\to 0}{=} A^i(\sigma)\left[1+O(\tau^2)\right] + \tau^{\frac{1}{k+1}}\, B^i(\sigma)\,\left[1+O(\tau^2)\right] + \tau^{2-\frac{1}{k+1}}\, C^i(\sigma)\,\left[1+O(\tau^2)\right]\;, \\ & 1 \leq i \leq D-1\;. \end{aligned} \tag{4.17}$$

where the residual conformal invariance (3.5) has been used.

The string equations of motion and constraints impose the following relations on the coefficients:

$$\begin{aligned} &\sum_{i=1}^{D-1}\left[B^i(\sigma)\right]^2 = 1 \quad , \quad \sum_{i=1}^{D-1} B^i(\sigma)\, A'^i(\sigma) = 0\;, \\ &C^i(\sigma) = -\eta_1(\sigma)\, B^i(\sigma) \quad , \quad \eta_1(\sigma) = \frac{(k+1)^2}{4(2k+1)} \sum_{i=1}^{D-1}\left[A'^i(\sigma)\right]^2\;. \end{aligned} \tag{4.18}$$

$A^i(\sigma),\ B^i(\sigma),\ 1 \leq i \leq D-1$ turn to be the initial ($\tau = 0$) string coordinates and momenta.

We again find that the string solution is determined by $2(D-2)$ independent functions indicating that only the transverse coordinates are physical. As we see, the τ-expansion produces for FRW universes the string solution as a series in powers of τ^2 and $\tau^{\frac{2k}{1+k}}$.

For large R the spacetime curvature tends to zero as $T^{-2} \simeq R^{-2/\alpha}$ in FRW spacetimes. That is, for $T \to \infty$. In order to analyze the string behaviour in such regime it is convenient to choose the gauge

$$\eta = \eta(\tau) \to \infty \quad , \quad \text{for } \tau \to \infty \tag{4.19}$$

[This is a slight generalization of the previous gauge choices at finite τ].

We then find from eq.(4.7)

$$\begin{aligned} \eta(\tau,\sigma) &\overset{\tau\to\infty}{=} \tau^{\frac{2}{k+2}} \to \infty\;, \\ X^i(\tau,\sigma) &\overset{\tau\to\infty}{=} \frac{1}{k+2}\,\tau^{-\frac{k}{k+2}}\left[f_i^+(\sigma+\tau) + f_i^-(\sigma-\tau)\right]\;, \\ & 1 \leq i \leq D-1\;. \end{aligned} \tag{4.20}$$

where the $f_i^\pm(x)$ are arbitrary periodic functions of x

$$f_i^\pm(x+2\pi) = f_i^\pm(x)\;,$$

obeying the pair of constraints:

$$\sum_{i=1}^{D-1} \left[f_i'^{\pm}(x) \right]^2 = 1 \; . \tag{4.21}$$

We obtain from eq.(4.20)

$$T(\tau) \overset{\tau \to \infty}{=} \frac{2\sqrt{A}}{k+2}\, \tau \to \infty \quad \text{and} \quad R \overset{\tau \to \infty}{=} \tau^{\frac{k}{k+2}} \; \to \infty \; . \tag{4.22}$$

In short, the string solutions in this regime are asymptotically Minkowski solutions (2.13) *scaled* by a factor R^{-1}. This is not unexpected since the spacetime curvature vanishes for this regime. The counting of degrees of freedom is again as in Minkowski spacetime.

We have determined the string behaviour for $R \to \infty$ in inflationary and FRW universes and for $R \to 0$ in FRW universes. Let us now compute for such regimes the string physical properties, energy-momentum and size.

The calculation of $\Theta^{AB}(T)$ in the different limiting regimes follows from eq.(2.29) since $\eta = \eta(\tau)$ asymptotically (both for $\tau \to 0$ and $\tau \to \infty$).

Let us start by considering the inflationary universes for $R \to \infty$. We find for the (integrated) energy-momentum tensor from eqs.(2.29) and (4.15)

$$\rho(T) \;=\; \Theta^{00}(T) \overset{\tau \to 0}{=} \frac{R}{\alpha'} \to +\infty \; , \tag{4.23}$$

$$\begin{aligned} \Theta^i_j(T) &\overset{\tau \to 0}{=} \frac{R}{\alpha'} \int_0^{2\pi} \frac{d\sigma}{2\pi}\, A'^i(\sigma)\, A'^j(\sigma) \; \to \infty \; , \\ \Theta^0_i(T) &\overset{\tau \to 0}{=} -\frac{1}{\alpha'} \int_0^{2\pi} \frac{d\sigma}{2\pi}\, B^i(\sigma) \end{aligned} \tag{4.24}$$

where we also used the spacetime metric $G_{00} = 1, \; G_{ij} = -R^2\, \delta_{ij}$.

Recall that $\sum_{i=1}^{D-1} [A'^i(\sigma)]^2 = 1$.

We see that the energy density $\rho(T)$ diverges for $R \to \infty$. The stress tensor $[\Theta^i_j(T)]$ is not diagonal but it is given by a **positive definite** matrix. Such matrix has then positive eigenvalues and therefore tells us that the pressure is **negative** [compare with eq.(4.8)]. This is the **unstable** string behaviour. That is, the energy tends to $+\infty$ and the pressure to $-\infty$. At the same time the energy flux density $\Theta^0_i(T)$ stands bounded.

The string size S in cosmological spacetimes (4.1) takes the form

$$S^2 = G_{AB}(X)\, \dot{X}^A\, \dot{X}^B = \dot{T}^2 - R^2 \sum_{i=1}^{D-1} (\dot{X}^i)^2 \tag{4.25}$$

For $\tau \to 0, \;\; R \to \infty$ in inflationary spacetimes, we find using eq.(4.15) that the first term dominates in eq.(4.25)

$$S \stackrel{\tau\to 0}{=} c\,\tau^{k/2} \simeq R \to \infty \;, \tag{4.26}$$

where c is a constant. We see that the string grows infinitely big when the universe inflates. The string size being proportional to the scale factor and also to the string energy [ρ in eq.(4.23)]. These explosive growings characterize the string unstable behaviour.

Let us now consider FRW universes ($k > 0$) for $\tau \to 0,\; R \to 0$. The (integrated) energy-momentum tensor in such regime takes the form

$$\rho(T) \;=\; \Theta^{00}(T) \stackrel{\tau\to 0}{=} \frac{1}{\alpha'(k+1)\;R} \to +\infty \;, \tag{4.27}$$

$$\Theta^i_j(T) \stackrel{\tau\to 0}{=} -\frac{1}{\alpha'(k+1)\;R} \int_0^{2\pi} \frac{d\sigma}{2\pi}\; B^i(\sigma)\; B^j(\sigma) \;\to \infty \;, \tag{4.28}$$

$$\Theta^0_i(T) \stackrel{\tau\to 0}{=} \frac{1}{\alpha'(k+1)\;R} \int_0^{2\pi} \frac{d\sigma}{2\pi}\; B^i(\sigma)$$

where we used eqs.(2.29) and (4.17). [Recall that $\sum_{i=1}^{D-1}\;[B^i(\sigma)]^2 = 1$].

We see in eq.(4.27) that the stress tensor ($\Theta^i_j(T)$) is not diagonal but it is given by a **negative definite** matrix. Such matrix has then negative eigenvalues and therefore tells us that the pressure is **positive**. This string behaviour is dual to the previous unstable behaviour.

We find from eq.(4.25) for the string size S in FRW universes for $R \to 0$,

$$S \stackrel{\tau\to 0}{=} \sqrt{\sum_{i=1}^{D-1} [A'^i(\sigma)]^2}\; \tau^{\frac{k}{2(k+1)}} \simeq \sqrt{\sum_{i=1}^{D-1} [A'^i(\sigma)]^2}\; R \to 0 \tag{4.29}$$

In this dual to unstable behaviour, the string size **vanishes**. That is, the string starts at the big bang with zero size.

In summary, the energy tends to $+\infty$, the pressure also to $+\infty$ and the size tends to zero in the **dual to unstable** behaviour.

In this regime strings behave as radiation (massless particles). Recall that the string size is proportional to the trace of the energy momentum tensor [see eq.(2.34)].

Let us finally consider FRW universes ($k > 0$) for $\tau \to \infty,\; R \to \infty$. There, the (integrated) energy-momentum tensor takes the form

$$\rho(T) \;=\; \Theta^{00}(T) \stackrel{\tau\to\infty}{=} \frac{2\sqrt{A}}{\alpha'(k+2)} \;, \tag{4.30}$$

$$\Theta^i_j(T) \stackrel{\tau\to\infty}{=} \frac{R}{\alpha'} \int_0^{2\pi} \frac{d\sigma}{2\pi}\; \left[f'^+_i\; f'^-_j + f'^-_i\; f'^+_j \right] \;, \tag{4.31}$$

$$\Theta^0_i(T) \stackrel{\tau\to\infty}{=} 0 \;.$$

where we used eqs.(2.29) and (4.20). The string energy here tends to a bounded constant. Since the $f'^{\pm}_i(\sigma \pm \tau)$ are periodic functions, their average on a period of time vanishes:

$$\int_0^{2\pi} d\tau d\sigma\; f'^+_i(\sigma+\tau)\; f'^-_i(\sigma-\tau) = \frac{1}{2}\int_{-2\pi}^{+2\pi} dx_- \int_{|x_-|}^{4\pi-|x_-|} dx_+\; f'^+_i(x_+)\; f'^-_i(x_-) = 0 \;.$$

Hence, the pressure vanishes when averaged over a string oscillation. This is the **stable** string behaviour. Here strings behave as dust (cold matter) with $p = 0$ as equation of state.

The string size S follows eq.(4.25) and eq.(4.20),

$$S^2 \overset{\tau\to\infty}{=} \frac{2}{(k+2)^2} \left[1 + \sum_{i=1}^{D-1} f_i'^+ f_i'^-\right] \tag{4.32}$$

The string size is thus bounded for $R \to \infty$. Moreover, averaging over a period of time, we find

$$\bar{S} = \frac{\sqrt{2}}{k+2} .$$

B. The perfect gas of strings

Our aim is to provide a string description of matter appropriate to the early universe.

Let us consider classical strings interacting with the cosmological spacetime background and neglect their mutual interactions. That is, we consider a perfect gas of strings under the cosmic gravitational field. The energy-momentum of such gas is just the sum over individual string solutions. For each string the results of section IV.A apply.

We assume arbitrary initial data for the strings. Therefore, summing over solutions is equivalent to *average* over the initial data $A^i(\sigma),\ B^i(\sigma),\ 1 \leq i \leq D-1$.

For inflationary spacetimes the relevant quantity to average is the integral

$$\int_0^{2\pi} \frac{d\sigma}{2\pi}\, A'^i(\sigma)\, A'^j(\sigma)$$

that appears in the energy-momentum tensor eq.(4.23).

The average will vanish for $i \neq j$ and will be i-independent for $i = j$ because we treat independently the different $A^i(\sigma)$. Taking into account the constraint (4.16)

$$\sum_{i=1}^{D-1} \left[A'^i(\sigma)\right]^2 = 1 ,$$

finally yields,

$$< \int_0^{2\pi} \frac{d\sigma}{2\pi}\, A'^i(\sigma)\, A'^j(\sigma) > = \frac{\delta_{ij}}{D-1} . \tag{4.33}$$

Therefore the stress tensor (4.23) takes for unstable strings the fluid form for $R \to \infty$,

$$< \Theta^i_j(T) > \overset{R\to\infty}{=} -\frac{p}{D-1}\, \delta_{ij} ,$$

with

$$p \equiv -\frac{R}{\alpha'(D-1)} = -\frac{\rho}{D-1} \to -\infty , \tag{4.34}$$

where we used the expression for ρ in eq.(4.23). Recall that the string size also grows as R for $R \to \infty$ [eq.(4.26)].

This equation of state exactly saturates the strong energy condition in general relativity.

The unstable string behaviour corresponds to the critical case of the so-called coasting universe [19,32]. In other words, the perfect gas of strings provide a *concrete* matter realization of such cosmological model. Till now, no form of matter was known to describe coasting universes [19].

The quantity to average in FRW spacetimes for small R is

$$\int_0^{2\pi} \frac{d\sigma}{2\pi}\, B^i(\sigma)\, B^j(\sigma)$$

which appears in the energy-momentum tensor eq.(4.27).

Following the same argument as above for the average over $A^i(\sigma)$ and recalling that (4.18)

$$\sum_{i=1}^{D-1} \left[B^i(\sigma)\right]^2 = 1 \; ,$$

we find

$$< \int_0^{2\pi} \frac{d\sigma}{2\pi}\, B^i(\sigma)\, B^j(\sigma) >= \frac{\delta_{ij}}{D-1} \tag{4.35}$$

Hence the string energy-momentum tensor (4.27) for dual to unstable strings takes the fluid form for $R \to 0$,

$$< \Theta^i_j(T) > \stackrel{R \to 0}{=} -\frac{p}{D-1}\, \delta_{ij}$$

with

$$p \equiv \frac{1}{\alpha'(D-1)(k+1)\, R} \; \stackrel{R \to 0}{=} +\frac{\rho}{D-1} \to +\infty$$

where we used the expression for ρ in eq.(4.27). Recall that the string size vanishes as R, as R vanishes [eq.(4.29)].

Therefore in the dual to unstable case, strings behave as radiation (massless particles).

For large R in FRW spacetimes we must average independently over the functions f_i^+ and f_j^-, $1 \leq i, j \leq D-1$. We find then from eqs.(4.30)

$$< \Theta^i_j(T) > \stackrel{R \to \infty}{=} 0 \; .$$

That is, the equation of state

$$p = 0 \; .$$

The string energy and size are bounded in this regime. The strings behave for the stable regime as dust (cold matter). That is, they behave as massive particles.

In conclusion, an ideal gas of classical strings in cosmological universes exhibit three different thermodynamical behaviours, all of perfect fluid type:

- (1) For inflationary universes and $R \to \infty$ unstable strings: negative pressure gas with $p = -\frac{\rho}{D-1}$.
- (2) Dual behaviour in FRW universes and $R \to 0$: positive pressure gas similar to radiation, $p = +\frac{\rho}{D-1}$.
- (3) Stable strings in FRW universes and $R \to \infty$: positive pressure gas similar to cold matter, $p = 0$.

Tables 1 and 2 summarize the main string properties for any scale factor $R(T)$.

TABLE 1. String energy and pressure as obtained from exact string solutions for various expansion factors $R(T)$.

STRING PROPERTIES FOR ARBITRARY $R(T)$

D-Dimensional spacetimes: three asymptotic behaviours (u, d, s)	Energy	Pressure	Equation of State:
(i) unstable for $R \to \infty$	$E_u \overset{R\to\infty}{=} u\, R \to \infty$	$P_u = -\frac{E_u}{D-1} \to -\infty$	'stringy'
(ii) dual to (i) for $R \to 0$	$E_d \overset{R\to 0}{=} d/R \to \infty$	$P_d = +\frac{E_d}{D-1} \to \infty$	radiation
(iii) stable for $R \to \infty$	$E_s =$ constant	$P_s = 0$	dust (cold matter)

TABLE 2. The string energy density and pressure for a gas of strings can be summarized by the formulas below which become exact for $R \to 0$ and for $R \to \infty$.

STRING ENERGY DENSITY AND PRESSURE FOR ARBITRARY $R(T)$

	Energy density: $\rho \equiv E/R^{D-1}$	Pressure
Qualitatively correct formulas for all R and D	$\rho = \left(u\, R + \frac{d}{R} + s\right) \frac{1}{R^{D-1}}$	$p = \frac{1}{D-1} \left(\frac{d}{R} - u\, R\right) \frac{1}{R^{D-1}}$

Finally, notice that strings continuously evolve from one type of behaviour to the other two. This is explicitly seen from the string solutions in refs. [5] - [7] . For example the string described by $q_-(\sigma,\tau)$ for $\tau > 0$ shows unstable behaviour for $\tau \to 0$, dual behaviour for $\tau \to \tau_0 = 1.246450...$ and stable behaviour for $\tau \to \infty$.

TABLE 3. The **self-consistent** cosmological solution of the Einstein equations in General Relativity with the string gas as source.

STRING COSMOLOGY IN GENERAL RELATIVITY

Einstein equations (no dilaton field)	Expansion factor $R(T)$	Temperature $T(R)$
$T \to 0$	$\frac{D}{2}\left[\frac{2d}{(D-1)(D-2)}\right]^{\frac{1}{D}} \; T^{\frac{2}{D}}$	$\frac{dD}{S(D-1)}\; 1/R$
$T \to \infty$	$\left[\frac{(D-1)s}{2(D-2)}\right]^{\frac{1}{D-1}} \; T^{\frac{2}{D-1}}$	usual matter dominated behaviour

V. SELF-CONSISTENT STRING COSMOLOGY

In the previous section we investigated the propagation of test strings in cosmological space-times. Let us now investigate how the Einstein equations in General Relativity and the effective equations of string theory (beta functions) can be verified **self-consistently** with our string solutions as sources.

We shall assume a gas of classical strings neglecting interactions as string splitting and coalescing. We will look for cosmological solutions described by metrics of the type (4.1). It is natural to assume that the background will have the same symmetry as the sources. That is, we assume that the string gas is homogeneous, described by a density energy $\rho = \rho(T)$ and a pressure $p = p(T)$. In the effective equations of string theory we consider a space independent dilaton field. Antisymmetric tensor fields wil be ignored.

A. String Dominated Universes in General Relativity (no dilaton field)

The Einstein equations for the geometry (4.1) take the form

$$\begin{aligned} \frac{1}{2}\,(D-1)(D-2)\,H^2 &= \rho \quad , \\ (D-2)\dot{H} + p + \rho &= 0 \quad . \end{aligned} \tag{5.1}$$

where $H \equiv \frac{dR}{dT}/R$. We know p and ρ as functions of R in asymptotic cases. For large R, the unstable strings dominate [eq.(4.34)] and we have for inflationary spacetimes

$$\rho = u\, R^{2-D} \quad , \quad p = -\frac{\rho}{D-1} \quad \text{for } R \to \infty \tag{5.2}$$

For small R, the dual regime dominates with

$$\rho = d\, R^{-D} \quad , \quad p = +\frac{\rho}{D-1} \quad \text{for } R \to 0 \tag{5.3}$$

We also know that stable solutions may be present with a contribution $\simeq R^{1-D}$ to ρ and with zero pressure. For intermediate values of R the form of ρ is clearly more complicated but a formula of the type

$$\rho = \left(u_R\, R + \frac{d}{R} + s\right) \frac{1}{R^{D-1}} \tag{5.4}$$

where

$$\lim_{R\to\infty} u_R = \begin{cases} 0 & \text{FRW} \\ u_\infty \neq 0 & \text{Inflationary} \end{cases} \tag{5.5}$$

This equation of state is qualitatively correct for all R and becomes exact for $R \to 0$ and $R \to \infty$. The parameters u_R, d and s are positive constants and the u_R varies smoothly with R.

The pressure associated to the energy density (5.4) takes then the form

$$p = \frac{1}{D-1}\left(\frac{d}{R} - u_R\, R\right) \frac{1}{R^{D-1}} \tag{5.6}$$

Inserting eq.(5.4) into the Einstein-Friedmann equations [eq.(5.1)] we find

$$\frac{1}{2}\,(D-1)(D-2)\,\left(\frac{dR}{dT}\right)^2 = \left(u_R\, R + \frac{d}{R} + s\right) \frac{1}{R^{D-3}} \tag{5.7}$$

We see that R is a monotonic function of the cosmic time T. Eq.(5.7) yields

$$T = \sqrt{\frac{(D-1)(D-2)}{2}}\, \int_0^R dR\, \frac{R^{D/2-1}}{\sqrt{u_R\, R^2 + d + s\, R}} \tag{5.8}$$

where we set $R(0) = 0$.

It is easy to derive the behavior of R for $T \to 0$ and for $T \to \infty$.

For $T \to 0$, $R \to 0$, the term d/R dominates in eq.(5.7) and

$$R(T) \overset{T\to 0}{\simeq} \frac{D}{2}\left[\frac{2d}{(D-1)(D-2)}\right]^{\frac{1}{D}} T^{\frac{2}{D}} \tag{5.9}$$

For $T \to \infty$, $R \to \infty$ and the term $u_R\, R$ dominates in eq.(5.7). Hence,

$$R(T) \stackrel{T\to\infty}{\simeq} \left[\frac{(D-2)u_\infty}{2(D-1)}\right]^{\frac{1}{D-2}} T^{\frac{2}{D-2}} \tag{5.10}$$

[u_R tends to a constant u_∞ for $R \to \infty$]. This expansion is faster than (cold) matter dominated universes where $R \simeq T^{\frac{2}{D-1}}$. For example, for $D = 4$, R grows linearly with T whereas for matter dominated universes $R \simeq T^{2/3}$. However, eq.(5.10) **is not** a self-consistent solution. Assuming that the term $u_R\,R$ dominates for large R we find a scale factor $R(T) \simeq T^{\frac{2}{D-2}} \simeq \eta^{\frac{2}{D-4}}$ for $D \neq 4$ and $R(T) \simeq T \simeq e^\eta$ at $D = 4$. This **is not an inflationary universe** but a FRW universe. The term $u_R R$ is absent for large R in FRW universes as explained before. Therefore, we must instead use for large R

$$\rho = \left(\frac{d}{R} + s\right)\frac{1}{R^{D-1}} \tag{5.11}$$

Now, for $T \to \infty, R \to \infty$ and we find a matter dominated regime:

$$R(T) \stackrel{T\to\infty}{\simeq} \left[\frac{(D-1)s}{2(D-2)}\right]^{\frac{1}{D-1}} T^{\frac{2}{D-1}} \tag{5.12}$$

For intermediate values of T , $R(T)$ is a continuous and monotonically increasing function of T.

In summary, the universe starts at $T = 0$ with a singularity of the type dominated by radiation. (The string behaviour for $R \to 0$ is like usual radiation). Then, the universe expands monotonically, growing for large T as $R \simeq T^{\frac{2}{D-1}}$. In particular, this gives $R \simeq T^{2/3}$ for $D = 4$.

It must be noticed that the qualitative form of the solution $R(T)$ does not depend on the particular positive values of u_R, d and s.

We want to stress that we achieve a **self-consistent** solution of the Einstein equations with string sources since the behaviour of the string pressure and density given by eqs.(5.4)-(5.6) precisely holds in universes with power like $R(T)$.

In ref. [3] similar results were derived using arguments based on the splitting of long strings.

B. Thermodynamics of strings in cosmological spacetimes

Let us consider a comoving volume R^{D-1} filled by a gas of strings. The entropy change for this system is given by:

$$\mathcal{T} dS = d(\rho\, R^{D-1}) \ + \ p\, d(R^{D-1}) \tag{5.13}$$

The continuity equation (4.9) and (5.13) implies that dS/dT vanishes. That is, the entropy per comoving volume stays constant in time. Using now the thermodynamic relation [25]

$$\frac{dp}{d\mathcal{T}} = \frac{p+\rho}{\mathcal{T}} \tag{5.14}$$

it follows [26] that

$$S = \frac{R^{D-1}}{\mathcal{T}}(p+\rho) + \text{constant} \tag{5.15}$$

Eq.(5.15) together with eqs.(5.4) and (5.6) yields the temperature as a function of the expansion factor R. That is,

$$\mathcal{T} = \frac{1}{S}\left\{ s + \frac{1}{D-1}\left[\frac{D\,d}{R} + (D-2)\,u_R\,R\right]\right\} \tag{5.16}$$

where S stands for the (constant) value of the entropy.

Eq.(5.16) shows that for small R, $\mathcal{T}$ scales as $1/R$ whereas for large R it scales as R. The small R behaviour of $\mathcal{T}$ is the usual exhibited by radiation.

For large R, in FRW universes $u_R \to 0$ and the constant term in s dominates. We just find a cold matter behaviour for large R.

For large R in inflationary universes, $u_R \to u_\infty$ and eq.(5.16) would indicate a temperature that **grows** proportionally to R. However, as stressed in ref. [3], the decay of long strings (through splitting) makes u_R exponentially decreasing with R.

VI. EFFECTIVE STRING EQUATIONS WITH THE STRING SOURCES INCLUDED

Let us consider now the cosmological equations obtained from the low energy string effective action including the string matter as a classical source. In D spacetime dimensions, this action can be written as

$$\begin{aligned}
S &= S_1 + S_2 \\
S_1 &= \frac{1}{2}\int d^Dx\,\sqrt{-G}\;e^{-\Phi}\;\Big[\,R + G_{AB}\,\partial^A\Phi\,\partial^B\Phi\;+2\,U(G,\Phi) - c\,\Big] \\
S_2 &= -\frac{1}{4\pi\alpha'}\sum_{strings}\int d\sigma d\tau\;G_{AB}(X)\;\partial_\mu X^A\;\partial^\mu X^B \qquad ,
\end{aligned} \tag{6.1}$$

Here $A, B = 0, \ldots, D-1$. This action is written in the so called 'Brans-Dicke frame' (BD) or 'string frame', in which matter couples to the metric tensor in the standard way. The BD frame metric coincides with the sigma model metric to which test strings are coupled.

Eq.(6.1) includes the dilaton field (Φ) with a potential $U(G,\Phi)$ depending on the dilaton and graviton backgrounds; c stands for the central charge deficit or cosmological constant term. The antisymmetric tensor field was not included, in fact it is irrelevant for the results obtained here. Extremizing the action (6.1) with respect to G_{AB} and Φ yields the equations of motion

$$\begin{aligned}
R_{AB} + \nabla_{AB}\Phi + 2\,\frac{\partial U}{\partial G_{AB}} - \frac{G_{AB}}{2}\Big[R + 2\,\nabla^2\Phi - (\nabla\Phi)^2 - c + 2\,U\Big] &= e^{\phi}\;T_{AB} \\
R + 2\,\nabla^2\Phi - (\nabla\Phi)^2 - c + 2U - \frac{\partial U}{\partial\Phi} &= 0\;,
\end{aligned} \tag{6.2}$$

which can be more simply combined as

$$R_{AB} + \nabla_{AB}\Phi + 2\,\frac{\partial U}{\partial G_{AB}} - G_{AB}\,\frac{\partial U}{\partial\Phi} = e^{\Phi}\;T_{AB}$$

$$R + 2\,\nabla^2\Phi - (\nabla\Phi)^2 - c + 2\,U - \frac{\partial U}{\partial \Phi} = 0 \tag{6.3}$$

Here T_{AB} stands for the energy momentum tensor of the strings as defined by eq.(2.27). It is also convenient to write these equations as

$$R_{AB} - \frac{G_{AB}}{2}\,R = T_{AB} + \tau_{AB} \tag{6.4}$$

where τ_{AB} is the dilaton energy momentum tensor :

$$\tau_{AB} = -\nabla_{AB}\Phi + \frac{G_{AB}}{2}\left[2\,\frac{\partial U}{\partial \Phi} - R\right]$$

The Bianchi identity

$$\nabla^A\left(R_{AB} - \frac{G_{AB}}{2}\,R\right) = 0$$

yields, as it must be, the conservation equation,

$$\nabla^A\left(T_{AB} + \tau_{AB}\right) = 0 \tag{6.5}$$

It must be noticed that eqs.(6.3) do not reduce to the Einstein equations of General Relativity even when $\Phi = U = 0$. Eqs. (6.3) yields in that case the Einstein equations *plus* the condition $R = 0$.

A. Effective String Equations in Cosmological Universes

For the homogeneous isotropic spacetime geometries described by eq.(4.1) we have

$$\begin{aligned} R^0_0 &= -(D-1)(\dot H + H^2) \\ R^k_i &= -\delta^k_i\,[\dot H + (D-1)\,H^2] \\ R &= -(D-1)(2\,\dot H + D\,H^2). \end{aligned} \tag{6.6}$$

where $H \equiv \frac{1}{R}\frac{dR}{dT}$.

The equations of motion (6.3) read

$$\begin{aligned} \ddot\Phi - (D-1)(\dot H + H^2) - \frac{\partial U}{\partial \Phi} &= e^{\Phi}\;\rho \\ \dot H + (D-1)\,H^2 - H\,\dot\Phi + \frac{\partial U}{\partial \Phi} + \frac{R}{D-1}\frac{\partial U}{\partial R} &= e^{\Phi}\;p \\ 2\,\ddot\Phi + 2(D-1)\,H\,\dot\Phi - \dot\Phi^2 - (D-1)(2\,\dot H + D\,H^2) - 2\,\frac{\partial U}{\partial \Phi} - c + 2\,U &= 0 \end{aligned} \tag{6.7}$$

where dot $\cdot$ stands for $\frac{d}{dT}$, and

$$\rho = T^0_0 \qquad , \qquad -\delta^k_i\;p = T^k_i\;. \tag{6.8}$$

The conservation equation takes the form of eq.(4.9)

$$\dot{\rho} + (D-1)\, H\, (p+\rho) = 0 \quad . \tag{6.9}$$

By defining,

$$\begin{aligned} &\Psi \equiv \Phi - \log\sqrt{-G} = \Phi - (D-1)\,\log R \\ &\bar{\rho} = e^{\Phi}\,\rho \quad , \quad \bar{p} = e^{\Phi}\,p\ , \end{aligned} \tag{6.10}$$

equations (6.7) can be expressed in a more compact form as

$$\begin{aligned} \ddot{\Psi} - (D-1)\,H^2 - \left.\frac{\partial U}{\partial \Psi}\right|_R &= \bar{\rho} \\ \dot{H} - H\,\dot{\Psi} + \frac{R}{D-1}\left.\frac{\partial U}{\partial R}\right|_\Psi &= \bar{p} \\ \dot{\Psi}^2 - (D-1)\,H^2 - 2\,\bar{\rho} - 2\,U + c = 0\ , \end{aligned} \tag{6.11}$$

The conservation equation reads

$$\dot{\bar{\rho}} - \dot{\Psi}\,\bar{\rho} + (D-1)\,H\,\bar{p} = 0 \tag{6.12}$$

As is known, under the duality transformation $R \longrightarrow R^{-1}$, the dilaton transforms as $\Phi \longrightarrow \Phi + (D-1)\,\log R$. The shifted dilaton Ψ defined by eq.(6.10) is invariant under duality.
The transformation

$$R' \equiv R^{-1} \quad , \tag{6.13}$$

implies

$$\Psi' = \Psi \quad , \; H' = -H \quad , \; \bar{p}' = -p \quad , \; \bar{\rho}' = \bar{\rho} \tag{6.14}$$

provided $u = d$, that is, a duality invariant string source. This is the duality invariance transformation of eqs.(6.11).

Solutions to the effective string equations have been extensively treated in the literature [28] and they are not our main purpose. For the sake of completeness, we briefly analyze the limiting behaviour of these equations for $R \to \infty$ and $R \to 0$.

It is difficult to make a complete analysis of the effective string equations (6.11) since the knowledge about the potential U is rather incomplete. For weak coupling (e^{Φ} small) the supersymmetry breaking produces an effective potential that decreases very fast (as the exponential of an exponential of Φ) for $\Phi \to -\infty$.

Let us analyze the asymptotic behavior of eqs.(6.11) for $R \to \infty$ and $R \to 0$ assuming that the potential U can be ignored. It is easy to see that a power behaviour Ansatz both for R and for e^{Ψ} as functions of T is consistent with these equations. It turns out that the string sources do not contribute to the leading behaviour here, and we find for $R \to 0$

$$R_{\mp} = C_1\,T^{\pm 1/\sqrt{D-1}} \;\to 0 \quad ,$$

$$e^{\Psi_\mp} = \quad C_2\, T^{-1} \;\rightarrow \begin{cases} \infty \\ 0 \end{cases} \tag{6.15}$$

Where C_1 and C_2 are constants. Here the branches $(-)$ and $(+)$ correspond to $T \rightarrow 0$ and to $T \rightarrow \infty$ respectively. In both regimes $R_\mp \rightarrow 0$ and $e^{\Phi_\mp} \rightarrow 0$.

The potential $U(\Phi)$ is hence negligible in these regimes. In terms of the conformal time η , the behaviours (6.15) result

$$\begin{aligned} R_\mp &= \quad C_1'\, \eta^{\pm\frac{1}{\sqrt{D-1}\mp 1}} \rightarrow 0 \\ e^{\Psi_\mp} &= C_2'\, \eta^{-\frac{\sqrt{D-1}}{\sqrt{D-1}\mp 1}} \rightarrow \begin{cases} \infty \\ 0 \end{cases} \end{aligned} \tag{6.16}$$

Where C_1' and C_2' are constants. The branch $(-)$ would describe an expanding non-inflationary behaviour near the initial singularity $T = 0$, while the branch $(+)$ describes a 'big crunch' situation and is rather unphysical.

Similarly, for $R \rightarrow \infty$ and $e^\Phi \rightarrow \infty$, we find

$$\begin{aligned} R_\mp &= D_1\, T^{\mp 1/\sqrt{D-1}} \;\rightarrow \infty \quad , \\ e^{\Psi_\mp} &= \quad D_2\, T^{-1} \;\rightarrow \begin{cases} \infty \\ 0 \end{cases} \end{aligned} \tag{6.17}$$

Where D_1 and D_2 are constants. Here again, the branches $(-)$ and $(+)$ correspond to $T \rightarrow 0$ and to $T \rightarrow \infty$ respectively, but now in both regimes $R_\mp \rightarrow \infty$ and $e^{\Phi_\mp} \rightarrow \infty$. (In this limit, one is not guaranteed that U can be consistently neglected). In terms of the conformal time, eqs.(6.17) read

$$\begin{aligned} R_\mp &= \quad D_1'\, \eta^{\mp\frac{1}{\sqrt{D-1}\pm 1}} \rightarrow \infty \\ e^{\Psi_\mp} &= D_2'\, \eta^{-\frac{\sqrt{D-1}}{\sqrt{D-1}\pm 1}} \rightarrow \begin{cases} \infty \\ 0 \end{cases} \end{aligned} \tag{6.18}$$

The branch $(+)$ describes a noninflationary expanding behaviour for $T \rightarrow \infty$ faster than the standard matter dominated expansion, while the branch $(-)$ describes a super-inflationary behaviour $\eta^{-\alpha}$, since $0 < \alpha < 1$, for all D.

The behaviours (6.15) for $R_\mp \rightarrow 0$ and (6.17) for $R_\mp \rightarrow \infty$ are related by duality $R \leftrightarrow 1/R$.

B. String driven inflation?

Let us consider now the question of whether de Sitter spacetime may be a self-consistent solution of the effective string equations (6.7) with the string sources included. The strings in cosmological universes like de Sitter spacetime have the equation of state (5.4)-(5.6). Since $e^\Psi = e^\Phi\, R^{1-D}$:

$$\bar{\rho} = e^\Psi \left(u\, R + \frac{d}{R} + s \right) \tag{6.19}$$

$$\bar{p} = \frac{e^\Psi}{D-1} \left(\frac{d}{R} - u\, R \right) \tag{6.20}$$

In the absence of dilaton potential and cosmological constant term, the string sources do not generate de Sitter spacetime as discussed in sec. V.A. We see that for $U = c = 0$, and $R = e^{HT}$, eqs.(6.11) yields to a contradiction (unless $D = 0$) for the value of Ψ , required to be $-HT+$ constant.

A self-consistent solution describing asymptotically de Sitter spacetime self-sustained by the string equation of state (6.19)-(6.20) is given by

$$\begin{aligned} R &= e^{HT} \quad , \quad H = \text{constant} > 0 \quad , \\ 2U - c &= D\ H^2 = \text{constant} \\ \Psi_{\pm} &= \mp HT \pm i\pi + \log \frac{(D-1)\,H^2}{\rho_{\pm}} \\ \rho_+ &\equiv u \quad , \quad \rho_- \equiv d \end{aligned} \tag{6.21}$$

The branch Ψ_+ describes the solution for $R \to \infty$ ($T \to +\infty$), while the branch Ψ_- corresponds to $R \to 0$ ($T \to -\infty$). De Sitter spacetime with lorentzian signature self-sustained by the strings necessarily requires a constant imaginary piece $\pm i\pi$ in the dilaton field. This makes $e^{\Psi} < 0$ telling us that the gravitational constant $G \sim e^{\Psi} < 0$ here describes antigravity.

Is interesting to notice that in the euclidean signature case, i. e. $(+++\ldots++)$, the Ansatz $\dot{H} = 0$, $2U - c$ =constant, yields a constant curvature geometry with a real dilaton, but which is of Anti-de Sitter type. This solution is obtained from eqs.(6.20)-(6.21) through the transformation

$$\hat{X}^0 = iT \quad , \quad \hat{H} = -iH \quad , \quad X^i = X^i \quad , \quad \Psi = \Psi \tag{6.22}$$

which maps the Lorentzian de Sitter metric into the positive definite one

$$d\hat{s}^2 = (d\hat{X}^0)^2 + e^{\hat{H}\hat{X}^0}\ (d\vec{X})^2. \tag{6.23}$$

The equations of motion (6.11) within the constant curvature Ansatz ($\dot{\hat{H}} = \ddot{\Psi} = 0$) are mapped onto the equations

$$\begin{aligned} (D-1)\,\hat{H}^2 - \left.\frac{\partial U}{\partial \Psi}\right|_R &= \bar{\rho} \\ \hat{H}\,\frac{d\Psi}{d\hat{X}^0} + \frac{R}{D-1}\left.\frac{\partial U}{\partial R}\right|_\Psi &= \bar{p} \\ -(\frac{d\Psi}{d\hat{X}^0})^2 + (D-1)\,\hat{H}^2 - 2\,\bar{\rho} - 2\,U + c &= 0\ , \end{aligned} \tag{6.24}$$

with the solution

$$\begin{aligned} R &= e^{\hat{H}\hat{X}^0} \quad , \quad \hat{H} = \text{constant} > 0 \quad , \\ c - 2\,U &= D\ \hat{H}^2 = \text{constant} \\ \Psi_{\pm} &= \mp \hat{H}\hat{X}^0 + \log \frac{(D-1)\,\hat{H}^2}{\rho_{\pm}} \\ \rho_+ &\equiv u \quad , \quad \rho_- \equiv d \end{aligned} \tag{6.25}$$

Both solutions (6.25) and (6.21) are mapped one into another through the transformation (6.22).

It could be recalled that in the context of (point particle) field theory, de Sitter spacetime (as well as anti-de Sitter) emerges as an exact selfconsistent solution of the semiclassical Einstein equations with the back reaction included [34] - [35]. (Semiclassical in this context, means that matter fields including the graviton are quantized to the one-loop level and coupled to the (c-number) gravity background through the expectation value of the energy-momentum tensor T_A^B . This expectation value is given by the trace anomaly: $< T_A^A >= \bar{\gamma}\ R^2$). On the other hand, the α' expansion of the effective string action admits anti-de Sitter spacetime (but not de Sitter) as a solution when the quadratic curvature corrections (in terms of the Gauss-Bonnet term) to the Einstein action are included [36]. It appears that the corrections to the anti-de Sitter constant curvature are qualitatively similar in the both cases, with α' playing the rôle of the trace anomaly parameter $\bar{\gamma}$ [35].

The fact that de Sitter inflation with true gravity $G \sim e^{\Psi} > 0$ does not emerge as a solution of the effective string equations does not mean that string theory excludes inflation. What means is that the effective string equations are not enough to get inflation. The effective string action is a low energy field theory approximation to string theory containing only the *massless* string modes (*massless* background fields).

The vacuum energy scales to start inflation (physical or true vacuum) are typically of the order of the Planck mass [26] - [27] where the effective string action approximation breaks down. One must consider the massive string modes (which are absent from the effective string action) in order to properly get the cosmological condensate yielding de Sitter inflation. We do not have at present the solution of such problem.

TABLE 4. Asymptotic solution of the string effective equations (including the dilaton).

EFFECTIVE STRING EQUATIONS SOLUTIONS IN COSMOLOGY

Effective String equations	$R(T) \to 0$ behaviour	$R(T) \to \infty$ behaviour
$T \to 0$	$\simeq T^{+1/\sqrt{D-1}}$	$\simeq T^{-1/\sqrt{D-1}}$
$T \to \infty$	$\simeq T^{-1/\sqrt{D-1}}$	$\simeq T^{+1/\sqrt{D-1}}$

VII. MULTI-STRINGS AND SOLITON METHODS IN DE SITTER UNIVERSE

Among the cosmological backgrounds, de Sitter spacetime occupies a special place. This is, in one hand relevant for inflation and on the other hand string propagation turns to be specially interesting there [2] - [8]. String unstability, in the sense that the string proper length grows indefinitely is particularly present in de Sitter. The string dynamics in de Sitter universe

is described by a generalized sinh-Gordon model with a potential unbounded from below [4]. The sinh-Gordon function $\alpha(\sigma,\tau)$ having a clear physical meaning : $H^{-1}e^{\alpha(\sigma,\tau)/2}$ determines the string proper length. Moreover the classical string equations of motion (plus the string constraints) turn to be integrable in de Sitter universe [4,5]. More precisely, they are equivalent to a non-linear sigma model on the grassmannian $SO(D,1)/O(D)$ with periodic boundary conditions (for closed strings). This sigma model has an associated linear system [37] and using it, one can show the presence of an infinite number of conserved quantities [38]. In addition, the string constraints imply a zero energy-momentum tensor and these constraints are compatible with the integrability.

The so-called dressing method [37] in soliton theory allows to construct solutions of non-linear classically integrable models using the associated linear system. In ref. [6] we systematically construct string solutions in three dimensional de Sitter spacetime. We start from a given exactly known solution of the string equations of motion and constraints in de Sitter [5] and then we "dress" it. The string solutions reported there indeed apply to cosmic strings in de Sitter spacetime as well.

The invariant interval in D-dimensional de Sitter space-time is given by

$$ds^2 = dT^2 \ - \ \exp[2HT] \ \sum_{i=1}^{D-1}(dX^i)^2. \tag{7.1}$$

Here T is the so called cosmic time. In terms of the conformal time η,

$$\eta \equiv -\frac{\exp[-HT]}{H} \quad , -\infty < \eta \leq 0 \ ,$$

the line element becomes

$$ds^2 = \frac{1}{H^2\eta^2}[d\eta^2 \ - \ \sum_{i=1}^{D-1}(dX^i)^2 \,] \ .$$

The de Sitter spacetime can be considered as a D-dimensional hyperboloid embedded in a D+1 dimensional flat Minkowski spacetime with coordinates $(q^0,...,q^D)$:

$$ds^2 = \frac{1}{H^2}[-(dq^0)^2 \ + \ \sum_{i=1}^{D}(dq^i)^2 \,] \tag{7.2}$$

where

$$\begin{aligned}
q^0 &= \ \sinh HT \ + \frac{H^2}{2}\exp[HT] \ \sum_{i=1}^{D-1}(X^i)^2 \quad , \\
q^1 &= \ \cosh HT \ - \frac{H^2}{2}\exp[HT] \ \sum_{i=1}^{D-1}(X^i)^2 \quad , \\
q^{i+1} &= \ H\exp[HT] \ X^i \quad , \quad 1 \leq i \leq D-1, \ \ -\infty < T, \ X^i < +\infty.
\end{aligned} \tag{7.3}$$

The complete de Sitter manifold is the hyperboloid

$$-(q^0)^2 + \sum_{i=1}^{D}(q^i)^2 = 1.$$

The coordinates (T, X^i) and (η, X^i) cover only the half of the de Sitter manifold $q^0 + q^1 > 0$.

We will consider a string propagating in this D-dimensional space-time.The string equations of motion (2.7) in the metric (7.2) take the form:

$$\partial_{+-}q + (\partial_+ q.\partial_- q)\, q = 0 \quad \text{with} \quad q.q = 1, \tag{7.4}$$

where . stands for the Lorentzian scalar product $a.b \equiv -a_0 b_0 + \sum_{i=1}^{D} a_i b_i$, $x_\pm \equiv \frac{1}{2}(\tau \pm \sigma)$ and $\partial_\pm q = \frac{\partial q}{\partial x_\pm}$. The string constraints (2.8) become for de Sitter universe

$$T_{\pm\pm} = \frac{\partial q}{\partial x_\pm}.\frac{\partial q}{\partial x_\pm} = 0. \tag{7.5}$$

Eqs.(7.4) describe a non compact O(D,1) non-linear sigma model in two dimensions. In addition, the (two dimensional) energy-momentum tensor is required to vanish by the constraints eqs.(7.5) . This system of non-linear partial differential equations can be reduced by choosing an appropriate basis for the string coordinates in the $(D+1)$-dimensional Minkowski space time $(q^0, ..., q^D)$ to a noncompact Toda model [4].

These equations can be rewritten in the form of a chiral field model on the Grassmanian $G_D = SO(D,1)/O(D)$. Indeed, any element $\mathbf{g} \in G_D$ can be parametrized with a real vector $q\rangle$ of the unit pseudolength

$$\mathbf{g} = 1 - 2|q\rangle\langle q|J, \quad \langle q|J|q\rangle = 1. \tag{7.6}$$

In terms of $\mathbf{g}$, the string equations (7.4) have the following form

$$2\,\mathbf{g}_{\xi\eta} - \mathbf{g}_\xi\,\mathbf{g}\,\mathbf{g}_\eta - \mathbf{g}_\eta\,\mathbf{g}\,\mathbf{g}_\xi = 0\ , \tag{7.7}$$

and the conformal constraints (7.5) become

$$\mathrm{tr}\,\mathbf{g}_\xi^2 = 0, \quad \mathrm{tr}\,\mathbf{g}_\eta^2 = 0\ . \tag{7.8}$$

The fact that $\mathbf{g} \in G_D$ implies that $\mathbf{g}$ is a real matrix with the following properties:

$$\mathbf{g} = J\mathbf{g}^{\mathrm{t}}J, \quad \mathbf{g}^2 = I, \quad tr\,\mathbf{g} = D-1\ , \quad \mathbf{g} \in SL(D+1, R). \tag{7.9}$$

These conditions are equivalent to the existence of the representation (7.6). Equation (7.7) is the compatibility condition for the following overdetermined linear system:

$$\Psi_\xi = \frac{U}{1-\lambda}\Psi, \quad \Psi_\eta = \frac{V}{1+\lambda}\Psi, \tag{7.10}$$

where

$$U = \mathbf{g}_\xi\,\mathbf{g}, \quad V = \mathbf{g}_\eta\,\mathbf{g}\ \ . \tag{7.11}$$

Or, in terms of the vector $q\rangle$

$$U = 2\,q_\xi\rangle\langle q\,J - 2\,q\rangle\langle q_\xi\,J,$$
$$V = 2\,q_\eta\rangle\langle q\,J - 2\,q\rangle\langle q_\eta\,J.$$

Eq. (7.6) can be easily inverted yielding q in terms of the matrix g:

$$q_0 = \sqrt{\frac{g_{00} - 1}{2}} \quad , \quad q_i = \sqrt{\frac{1 - g_{ii}}{2}} \;\; 1 \leq i \leq D \text{ (no sum over } i) \tag{7.12}$$

The use of overdetermined linear systems to solve non-linear partial differential equations associated to them goes back to refs. [39]. (See refs. [40] - [41] for further references).

In order to fix the freedom in the definition of Ψ we shall identify

$$\Psi(\lambda = 0) = \mathbf{g}. \tag{7.13}$$

This condition is compatible with the above equations since the matrix function Ψ at the point $\lambda = 0$ satisfies the same equations as $\mathbf{g}$. Thus the problem of constructing exact solutions of the string equations is reduced to finding compatible solutions of the linear equations (7.10) such that $\mathbf{g} = \Psi(\lambda = 0)$ satisfies the constraints eqs.(7.8) and (7.9).

We concentrate below on the linear system (7.10) since this is the main tool to derive new string solutions in de Sitter spacetime.

In ref. [6] the dressing method was applied as follows. We started from the exact ring-shaped string solution $q_{(0)}$ [5] and we find the explicit solution $\Psi^{(0)}(\lambda)$ of the associated linear system, where λ stands for the spectral parameter. Then, we propose a new solution $\Psi(\lambda)$ that differs from $\Psi^{(0)}(\lambda)$ by a matrix rational in λ. Notice that $\Psi(\lambda = 0)$ provides in general a new string solution.

We then show that this rational matrix must have at least **four** poles, $\lambda_0, 1/\lambda_0, \lambda_0^*, 1/\lambda_0^*$, as a consequence of the symmetries of the problem. The residues of these poles are shown to be one-dimensional projectors. We then prove that these projectors are formed by vectors which can all be expressed in terms of an arbitrary complex constant vector $|x_0\rangle$ and the complex parameter λ_0. This result holds for arbitrary starting solutions $q_{(0)}$.

Since we consider closed strings, we impose a 2π-periodicity on the string variable σ . This restricts λ_0 to take discrete values that we succeed to express in terms of Pythagorean numbers. In summary, our solutions depend on two arbitrary complex numbers contained in $|x_0\rangle$ and two integers n and m . The counting of degrees of freedom is analogous to 2+1 Minkowski spacetime except that left and right modes are here mixed up in a non-linear and precise way.

The vector $|x_0\rangle$ somehow indicates the polarization of the string. The integers (n, m) determine the string winding. They fix the way in which the string winds around the origin in the spatial dimensions (here S^2). Our starting solution $q_{(0)}(\sigma, \tau)$ is a stable string winded $n^2 + m^2$ times around the origin in de Sitter space.

The matrix multiplications involved in the computation of the final solution were done with the help of the computer program of symbolic calculation "Mathematica". The resulting solution $q(\sigma, \tau) = (q^0, q^1, q^2, q^3)$ is a complicated combination of trigonometric functions of σ and hyperbolic functions of τ. That is, these string solitonic solutions do not oscillate in time. This is a typical feature of string unstability [5] - [11] - [9]. The new feature here is that strings (even stable solutions) do not oscillate neither for $\tau \to 0$, nor for $\tau \to \pm\infty$.

We plot in figs. 1-7 the solutions for significative values of $|x_0\rangle$ and (m, n) in terms of the comoving coordinates (T, X^1, X^2)

$$T = \frac{1}{H} \log(q^0 + q^1) \quad , \quad X^1 = \frac{1}{H} \frac{q^2}{q^0 + q^1} \quad , \quad X^2 = \frac{1}{H} \frac{q^3}{q^0 + q^1} \tag{7.14}$$

The first feature to point out is that our solitonic solutions describe **multiple** (here five or three) strings, as it can be seen from the fact that for a given time T we find several different values for τ. That is, τ is a **multivalued** function of T for any fixed σ (fig.1-2). Each branch of τ as a function of T corresponds to a different string. This is a entirely new feature for strings in curved spacetime, with no analogue in flat spacetime where the time coordinate can always be chosen proportional to τ. In flat spacetime, multiple string solutions are described by multiple world-sheets. Here, we have a **single** world-sheet describing several independent and simultaneous strings as a consequence of the coupling with the spacetime geometry. Notice that we consider *free* strings. (Interactions among the strings as splitting or merging are not considered). Five is the generic number of strings in our dressed solutions. The value five can be related to the fact that we are dressing a one-string solution ($q_{(0)}$) with *four* poles. Each pole adds here an unstable string.

In order to describe the real physical evolution, we eliminated numerically $\tau = \tau(\sigma, T)$ from the solution and expressed the spatial comoving coordinates X^1 and X^2 in terms of T and σ.

We plot $\tau(\sigma, T)$ as a function of σ for different fixed values of T in fig.3-4. It is a sinusoidal-type function. Besides the customary closed string period 2π, another period appears which varies on τ. For small τ , $\tau = \tau(\sigma, T)$ has a convoluted shape while for larger τ (here $\tau \leq 5$), it becomes a regular sinusoid. These behaviours reflect very clearly in the evolution of the spatial coordinates and shape of the string.

The evolution of the five (and three) strings simultaneously described by our solution as a function of T, for positive T is shown in figs. 5-7. One string is stable (the 5th one). The other four are unstable. For the stable string, (X^1, X^2) contracts in time precisely as e^{-HT}, thus keeping the proper amplitude $(e^{HT}X^1, e^{HT}X^2)$ and proper size constant. For this stable string $(X^1, X^2) \leq \frac{1}{H}$. ($1/H =$ the horizon radius). For the other (unstable) strings, (X^1, X^2) become very fast constant in time, the proper size expanding as the universe itself like e^{HT} . For these strings $(X^1, X^2) \geq \frac{1}{H}$. These exact solutions display remarkably the asymptotic string behaviour found in refs. [4,11].

In terms of the sinh-Gordon description, this means that for the strings outside the horizon the sinh-Gordon function $\alpha(\sigma, \tau)$ is the same as the cosmic time T up to a function of σ. More precisely,

$$\alpha(\sigma, \tau) \overset{T>\geq\frac{1}{H}}{=} 2H\, T(\sigma, \tau) + \log\left\{2H^2\left[(A^1(\sigma)')^2 + (A^2(\sigma)')^2\right]\right\} + O(e^{-2HT}). \qquad (7.15)$$

Here $A^1(\sigma)$ and $A^2(\sigma)$ are the X^1 and X^2 coordinates outside the horizon. For $T \to +\infty$ these strings are at the absolute *minimum* $\alpha = +\infty$ of the sinh-Gordon potential with infinite size. The string inside the horizon (stable string) corresponds to the *maximum* of the potential, $\alpha = 0$. $\alpha = 0$ is the only value in which the string can stay without being pushed down by the potential to $\alpha = \pm\infty$ and this also explains why only one stable string appears (is not possible to put more than one string at the maximum of the potential without falling down). These features are *generically* exhibited by our one-soliton multistring solutions, independently of the particular initial state of the string (fixed by $|x^0>$ and (n, m)). For particular values of $|x^0>$, the solution describes three strings, with symmetric shapes from $T = 0$, for instance like a rosette or a circle with festoons (fig. 5-7).

The string solutions presented here trivially embedd on D-dimensional de Sitter spacetime ($D \geq 3$). It must be noticed that they exhibit the essential physics of strings in D-dimensional

de Sitter universe. Moreover, the construction method used here works in any number of dimensions.

New classes of multistring solutions in curved spacetime has been recently found in [14].

VIII. STRINGS NEXT TO AND INSIDE BLACK HOLES

The classical string equations of motion and constraints were solved near the horizon and near the singularity of a Schwarzschild black hole in ref. [18]. Similar results have been obtained recently in ref. [23] using the null string approach [22].

In a conformal gauge such that $\tau = 0$ ($\tau =$ worldsheet time coordinate) corresponds to the horizon ($r = 1$) or to the black hole singularity ($r = 0$), the string coordinates express in power series in τ near the horizon and in power series in $\tau^{1/5}$ around $r = 0$.

In ref. [18] the string invariant size and the string energy-momentum tensor were computed. Near the horizon both are finite and analytic. Near the black hole singularity, the string size, the string energy and the transverse pressures (in the angular directions) tend to infinity as r^{-1}. To leading order near $r = 0$, the string behaves as two dimensional radiation. This two spatial dimensions are describing the S^2 sphere in the Schwarzschild manifold.

A. String Equations of motion in a Schwarzschild Black Hole.

The Schwarzschild metric in Schwarzschild coordinates (t, r, θ, ϕ) takes the following form:

$$ds^2 = \left(1 - \frac{1}{r}\right) \mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - \frac{1}{r}} - r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)\,, \tag{8.1}$$

where we choose units where the Schwarzschild radius $R_s = 2m = 1$.

Since we are interested in the whole Schwarzschild manifold and not just in the external part $r > 1$ where the static Schwarzschild coordinates are appropriate, we consider the Kruskal-Szekeres coordinates (u, v, θ, ϕ) defined by

$$u = t_K - r_K \equiv \sqrt{1-r}\; e^{(r-t)/2} \quad , \quad v = t_K + r_K \equiv \sqrt{1-r}\; e^{(r+t)/2}\,. \tag{8.2}$$

for $v \geq 0, u \geq 0$ and by

$$u = t_K - r_K \equiv -\sqrt{r-1}\; e^{(r-t)/2} \quad , \quad v = t_K + r_K \equiv \sqrt{r-1}\; e^{(r+t)/2}\,. \tag{8.3}$$

for $v \geq 0, u \leq 0$. For $v \leq 0$ one just flips the sign of v in eq.(8.2) or (8.3) [43].

The coordinate t_K is a time-like coordinate, and r_K spacelike. In Kruskal-Szekeres coordinates the Schwarzschild metric takes the form,

$$ds^2 = -\frac{4}{r}\; e^{-r}\; du\, dv + r^2\; (\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)\,. \tag{8.4}$$

r is a function of the product uv defined by the inverse function of

$$uv = [1 - r]\, e^r\;.$$

for $uv \geq 0$. The metric is such coordinates is regular everywhere except at its singularity, $r = 0$.

The string equations of motion in Schwarzschild coordinates and in the conformal gauge, are

$$\begin{aligned} r_\sigma t_\sigma - r_\tau t_\tau + r(r-1)(t_{\sigma\sigma} - t_{\tau\tau}) = 0\,, \\ \frac{2r}{1-r}(r_{\tau\tau} - r_{\sigma\sigma}) - \frac{1}{r^2}(t_\tau^2 - t_\sigma^2) + 2r(\theta_\tau^2 - \theta_\sigma^2) + \\ 2\,r\sin^2\theta\,(\phi_\tau^2 - \phi_\sigma^2) + \frac{1}{(r-1)^2}(r_\tau^2 - r_\sigma^2) = 0\,. \end{aligned} \tag{8.5}$$

$$\begin{aligned} r\sin\theta\,(\phi_{\tau\tau} - \phi_{\sigma\sigma}) + 2\,r\cos\theta\,(\phi_\tau\theta_\tau - \phi_\sigma\theta_\sigma) + 2\,\sin\theta\,(r_\tau\phi_\tau - r_\sigma\phi_\sigma) = 0\,, \\ r\,(\theta_{\tau\tau} - \theta_{\sigma\sigma}) + 2(r_\tau\theta_\tau - r_\sigma\theta_\sigma) - r\sin\theta\,\cos\theta\,(\phi_\tau^2 - \phi_\sigma^2) \qquad = 0\,. \end{aligned} \tag{8.6}$$

The constraints in Schwarzschild coordinates are

$$\begin{aligned} \frac{1-r}{r}(t_\sigma^2 + t_\tau^2) + \frac{r}{r-1}(r_\tau^2 + r_\sigma^2) + r^2\left[\theta_\tau^2 + \theta_\sigma^2 + \sin^2\theta\,(\phi_\tau^2 + \phi_\sigma^2)\right] = 0\,, \\ \frac{1-r}{r}t_\tau t_\sigma + \frac{r}{r-1}r_\tau r_\sigma + r^2\left(\theta_\tau\theta_\sigma + \sin^2\theta\,\phi_\tau\phi_\sigma\right) = 0\,. \end{aligned} \tag{8.7}$$

The string equations of motion in Kruskal-Szekeres coordinates take the form (always in the conformal gauge),

$$\begin{aligned} u_{\tau\tau} - u_{\sigma\sigma} + \frac{1}{r}\left(1 + \frac{1}{r}\right)\,e^{-r}\,v\,\left[(u_\tau)^2 - (u_\sigma)^2\right] - \frac{r\,u}{2}\left[\theta_\tau^2 - \theta_\sigma^2 + \sin^2\theta\,(\phi_\tau^2 - \phi_\sigma^2)\right] = 0 \\ v_{\tau\tau} - v_{\sigma\sigma} + \frac{1}{r}\left(1 + \frac{1}{r}\right)\,e^{-r}\,u\,\left[(v_\tau)^2 - (v_\sigma)^2\right] - \frac{r\,v}{2}\left[\theta_\tau^2 - \theta_\sigma^2 + \sin^2\theta\,(\phi_\tau^2 - \phi_\sigma^2)\right] = 0\,, \end{aligned} \tag{8.8}$$

plus eqs.(8.6) for the angular coordinates.

The constraints in Kruskal-Szekeres coordinates are

$$\begin{aligned} -\frac{4}{r}\,e^{-r}\,(u_\sigma\,v_\sigma + u_\tau\,v_\tau) + r^2\left[\theta_\tau^2 + \theta_\sigma^2 + \sin^2\theta\,(\phi_\tau^2 + \phi_\sigma^2)\right] = 0\,, \\ -\frac{4}{r}\,e^{-r}\,(u_\tau\,v_\sigma + u_\sigma\,v_\tau) + r^2\left(\theta_\tau\theta_\sigma + \sin^2\theta\,\phi_\tau\phi_\sigma\right) = 0\,. \end{aligned} \tag{8.9}$$

Notice that both the equations of motion and constraints are invariant under the exchange $u \leftrightarrow v$.

Also notice that the equations of motion and constraints in Kruskal-Szekeres coordinates are regular everywhere except at the singularity $r = 0$.

We shall consider closed strings where the string coordinates must be periodic functions of σ:

$$u(\sigma + 2\pi, \tau) = u(\sigma, \tau)\ ,\ v(\sigma + 2\pi, \tau) = v(\sigma, \tau). \tag{8.10}$$

Therefore, the angular coordinates θ, ϕ may be just quasiperiodic functions of σ:

$$\theta(\sigma + 2\pi, \tau) = \theta(\sigma, \tau) + \mathrm{mod}\,2\pi\ ,\ \phi(\sigma + 2\pi, \tau) = \phi(\sigma, \tau) + 2n\pi, \tag{8.11}$$

where n is an integer.

B. Strings Near the Singularity $r = 0$

Let us consider the solution of eqs.(8.6,8.8) and constraints (8.9), near $r = 0$. That is to say, near $uv = 1$.

For a generic world-sheet, we choose the gauge such that $\tau = 0$ corresponds to the string at the singularity $uv = 1$. This can be achieved as shown in general in sec. III.A.

Near the singularity $uv = 1$, we propose for $\tau \to 0$ the expansion [18]

$$\begin{aligned}
u(\sigma,\tau) &= e^{a(\sigma)}\left[1 - \tau^{\alpha}\,\beta(\sigma) + \ldots\right] \\
v(\sigma,\tau) &= e^{-a(\sigma)}\left[1 - \tau^{\alpha'}\,\hat{\beta}(\sigma) + \ldots\right] \\
\theta(\sigma,\tau) &= g(\sigma) + \tau^{\lambda'}\,\mu(\sigma) + \ldots, \\
\phi(\sigma,\tau) &= f(\sigma) + \tau^{\lambda}\,\nu(\sigma) + \ldots\,.
\end{aligned} \tag{8.12}$$

Inserting eqs.(8.12) in eqs.(8.6,8.8) and constraints (8.9) yields [18]

$$\begin{aligned}
&\alpha = \alpha' = 4/5 \quad , \quad \lambda = \lambda' = 1/5, \\
&\hat{\beta}(\sigma) = \beta(\sigma) \quad , \quad \beta(\sigma) = \frac{1}{64}\left[\mu(\sigma)^2 + \nu(\sigma)^2\,\sin^2 g(\sigma)\right]^2 .
\end{aligned} \tag{8.13}$$

Since the function $\beta(\sigma)$ is clearly positive, we write it as

$$\beta(\sigma) = \frac{1}{4}\,\gamma(\sigma)^4.$$

The coordinate r then vanishes as

$$r(\sigma,\tau) = \gamma(\sigma)^2\,\tau^{2/5} + \ldots\,. \tag{8.14}$$

The string solution is completely fixed once the functions $f(\sigma), g(\sigma), a(\sigma), \mu(\sigma)$ and $\nu(\sigma)$ are given. These five functions are arbitrary and can be expressed in terms of the initial data.

Notice that ϕ and θ approach their limiting values with the same exponent $1/5$ in τ.

Both the equations of motion and constraints are invariant under the exchange $u \leftrightarrow v$ but not the boundary conditions at $\tau = 0$. They differ by $a(\sigma) \leftrightarrow -a(\sigma)$ as we see from eqs.(8.12). Therefore one can obtain $u(\sigma,\tau)$ from $v(\sigma,\tau)$ and viceversa just by flipping the sign of $a(\sigma)$.

We can also find the ring solution of ref. [12] setting $f(\sigma) \equiv n\sigma, a(\sigma) \equiv 0, \mu(\sigma) =$ cte. $g(\sigma) =$ cte. and $\nu(\sigma) \equiv 0$ [see section VIII.D].

The corrections to the leading behaviour appear as positive integer powers of $\tau^{2/5}$. The subdominant leading power in $u(\sigma,\tau)$ and $v(\sigma,\tau)$ is again $\tau^{7/5}$. We find with the help of Mathematica [18],

$$\begin{aligned}
u(\sigma,\tau) = e^{a(\sigma)}\Big\{1 &- \gamma(\sigma)^4\,\tau^{4/5}\left[1 + O(\tau^{2/5})\right] \\
&- \gamma(\sigma)^6\,\frac{f'(\sigma)\nu(\sigma)\sin^2 g(\sigma) + \mu(\sigma)}{28\,a'(\sigma)}\,\tau^{7/5}\left[1 + O(\tau^{2/5})\right]\Big\}\,, \\
v(\sigma,\tau) = e^{-a(\sigma)}\Big\{1 &- \gamma(\sigma)^4\,\tau^{4/5}\left[1 + O(\tau^{2/5})\right]
\end{aligned}$$

$$+ \ \gamma(\sigma)^6 \ \frac{f'(\sigma)\nu(\sigma)\sin^2 g(\sigma) + \mu(\sigma)}{28\, a'(\sigma)} \ \tau^{7/5} \left[1 + O(\tau^{2/5})\right] \Bigg\} \ . \tag{8.15}$$

Notice that u/v is τ independent up to order $\tau^{7/5}$. Since $u/v = e^{-t}$, this imply that the spatial coordinate t is only σ-dependent up to $O(\tau^{7/5})$. More precisely,

$$t(\sigma,\tau) = \log\frac{v}{u} = -2\,a(\sigma) + \ \gamma(\sigma)^6 \ \frac{f'(\sigma)\nu(\sigma)\sin^2 g(\sigma) + \mu(\sigma)}{14\ a'(\sigma)} \ \tau^{7/5} + O(\tau^{9/5}) \ . \tag{8.16}$$

In other words, $t(\sigma,\tau)$ varies slower than the other coordinates ϕ and r when the string approaches the black hole singularity ($\tau \to 0$).

Using the diagonal conformal transformation (3.6), we can fix one of the arbitrary functions among $f(\sigma), g(\sigma), a(\sigma), \mu(\sigma)$ and $\nu(\sigma)$ keeping in mind the periodic boundary conditions:

$$a(\sigma+2\pi) = a(\sigma) \ , \ \nu(\sigma+2\pi) = \nu(\sigma) \ , \ \mu(\sigma+2\pi) = \mu(\sigma) \ ,$$

$$f(\sigma+2\pi) = f(\sigma) + 2n\pi \ , \ g(\sigma+2\pi) = g(\sigma) \bmod 2\pi \ . \tag{8.17}$$

We are left with **four** arbitrary functions of σ. This is precisely the number of transverse string degrees of freedom.

C. String energy-momentum and invariant size near the singularity

The string size in the Schwarzschild metric takes the form

$$S^2 = G_{AB}(X)\,\dot X^A\,\dot X^B = \frac{4}{r}\,e^{-r}\dot u\dot v - r^2\,\dot\theta^2 - r^2\,\dot\phi^2\,\sin^2\theta. \tag{8.18}$$

where we used eqs.(2.32) and (8.1).

We find near the singularity at $r = 0$ using eqs.(8.14-8.15)

$$\begin{aligned} S &= \frac{4\,a'(\sigma)^2}{\gamma(\sigma)^2}\ \tau^{-2/5} - \frac{4}{7}\ a'(\sigma)^2\ \left(6 + 25\ \frac{a'(\sigma)^2}{\gamma(\sigma)^7}\right) + O(\tau^{2/5}) \\ &= \frac{4\,a'(\sigma)^2}{r} - \frac{4}{7}\ a'(\sigma)^2\ \left(6 + 25\ \frac{a'(\sigma)^2}{\gamma(\sigma)^7}\right) + O(r)\ . \end{aligned} \tag{8.19}$$

For simplicity we choose here an equatorial solution at $\theta = \pi/2$. The invariant string size tends then to infinite when the string falls into the $r = 0$ singularity [18,23]. This is due to the infinitely growing gravitational forces that act there on the string.

The string stretching near $r = 0$ was first observed in ref. [42] using perturbative methods and in ref. [14] for a family of exact string solutions inside the horizon.

Inside the horizon we can use t, θ, ϕ as spatial coordinates and r as a coordinate time. We find,

$$\sqrt{g_{rr}}\,\Theta^{AB}(r) = \frac{1}{2\pi\alpha'}\int d\sigma d\tau\left(\dot X^A\dot X^B - X'^A X'^B\right)\delta(r - r(\tau,\sigma)). \tag{8.20}$$

where $g_{rr} = r/(1-r) > 0$.

We have for the black hole case:

$$G_{AB}(X)\, \dot{X}^A \dot{X}^B = \frac{r\dot{r}^2}{1-r} - r^2\,\dot{\theta}^2 - r^2\,\dot{\phi}^2\,\sin^2\theta - \frac{1-r}{r}\,\dot{t}^2\ . \tag{8.21}$$

Using eqs.(8.12) and (8.14) for $\tau \to 0$, we find that each of the first three terms grows as $\tau^{-4/5}$ whereas the last term vanishes as $\tau^{2/5}$. Moreover, the sum of the three terms $O(\tau^{-4/5})$ identically vanishes thanks to eq.(8.13). This cancellation in the trace tells us that near $r = 0$, the dominant (and divergent) components T^r_r, T^ϕ_ϕ and T^θ_θ yield a zero trace. This means that the string behaves to leading order as **two**-dimensional massless particles [18]. This is the so-called dual to unstable behaviour [1] (here for two spatial dimension).

For $\tau \to 0,\ r \to 0$ we can use in eq.(8.20) the dominant behaviours:

$$\begin{aligned}
r(\sigma,\tau) &= \gamma(\sigma)^2\,\tau^{2/5} + O(\tau^{4/5}),\\
\theta(\sigma,\tau) &= g(\sigma) + \mu(\sigma)\,\tau^{1/5} + O(\tau^{3/5}),\\
\phi(\sigma,\tau) &= f(\sigma) +\ \nu(\sigma)\,\tau^{1/5} + O(\tau^{3/5}),\\
t(\sigma,\tau) &= -2a(\sigma) + \gamma(\sigma)^6\ \frac{f'(\sigma)\nu(\sigma)\sin^2 g(\sigma) + \mu(\sigma)}{14\ a'(\sigma)}\ \tau^{7/5} + O\left(\tau^{9/5}\right)\ .
\end{aligned} \tag{8.22}$$

We thus find for $r \to 0$,

$$\begin{aligned}
2\pi\alpha'\,\Theta^{rr}(r) &= \frac{2}{5\ r^2}\int_0^{2\pi} \mathrm{d}\sigma\,|\gamma(\sigma)|^5 + O(\frac{1}{r}) \to +\infty\,,\\
2\pi\alpha'\,\Theta^{\phi\phi}(r) &= \frac{1}{10\ r^3}\int_0^{2\pi} \mathrm{d}\sigma\,\nu(\sigma)^2\ |\gamma(\sigma)|^3 + O(\frac{1}{r^2}) \to +\infty\,,\\
2\pi\alpha'\,\Theta^{\theta\theta}(r) &= \frac{1}{10\ r^3}\int_0^{2\pi} \mathrm{d}\sigma\,\mu(\sigma)^2\ |\gamma(\sigma)|^3 + O(\frac{1}{r^2}) \to +\infty\,,\\
2\pi\alpha'\,\Theta^{tt}(r) &= -10\ r\int_0^{2\pi} \mathrm{d}\sigma\ \frac{[a'(\sigma)]^2}{|\gamma(\sigma)|^5} + O(r^2) \to 0^-\ .
\end{aligned} \tag{8.23}$$

We can identify the string energy with the mixed component $-\Theta^r_r$. We define the mixed components $\Theta^B_A(r)$ by integrating $T^B_A(X)$ over the spatial volume.

This yields for $r \to 0$,

$$E \equiv -\Theta^r_r = \frac{1}{2\pi\alpha'}\ \frac{2}{5\ r}\int_0^{2\pi} \mathrm{d}\sigma\,|\gamma(\sigma)|^5 + O(1) \to +\infty\ . \tag{8.24}$$

The transverse pressures are defined as the mixed components Θ^ϕ_ϕ and Θ^θ_θ. They diverge for $r \to 0$:

$$\begin{aligned}
P_\phi \equiv \Theta^\phi_\phi &= \frac{1}{2\pi\alpha'}\ \frac{2}{5\ r}\int_0^{2\pi} \mathrm{d}\sigma\,\nu(\sigma)^2\ \sin^2 g(\sigma)\ |\gamma(\sigma)|^5\ \to +\infty\,,\\
P_\theta \equiv \Theta^\theta_\theta &= \frac{1}{2\pi\alpha'}\ \frac{2}{5\ r}\int_0^{2\pi} \mathrm{d}\sigma\,\mu(\sigma)^2\ |\gamma(\sigma)|^5\ \to +\infty.
\end{aligned} \tag{8.25}$$

Thus, to leading order,

$$E = P_\theta + P_\phi \quad \text{for} \quad r \to 0 \; .$$

exhibiting a two-dimensional ultrarelativistic gas behaviour. The tidal forces infinitely stretch the string near $r = 0$ in effectively only two directions: ϕ and θ.

We find for the off-diagonal components,

$$\begin{aligned}
2\pi\alpha'\,\Theta^{tr}(r) &= \frac{r^{1/2}}{10}\int_0^{2\pi}\frac{d\sigma}{a'(\sigma)}\,\gamma(\sigma)^4\,\left[f'(\sigma)\nu(\sigma)\sin^2 g(\sigma)+\mu(\sigma)\right]\to 0^+ \; ,\\
2\pi\alpha'\,\Theta^{t\theta}(r) &= \frac{2}{5}\int_0^{2\pi} d\sigma\,\frac{\mu(\sigma)}{a'(\sigma)}\,|\gamma(\sigma)|^3\,\left[f'(\sigma)\nu(\sigma)\sin^2 g(\sigma)+\mu(\sigma)\right]=O(1) \; ,\\
2\pi\alpha'\,\Theta^{t\phi}(r) &= \frac{2}{5}\int_0^{2\pi} d\sigma\,\frac{\nu(\sigma)}{a'(\sigma)}\,|\gamma(\sigma)|^3\,\left[f'(\sigma)\nu(\sigma)\sin^2 g(\sigma)+\mu(\sigma)\right]=O(1) \; ,\\
2\pi\alpha'\,\Theta^{r\phi}(r) &= \frac{1}{5\,r^{5/2}}\int_0^{2\pi} d\sigma\,\nu(\sigma)\,\gamma(\sigma)^4 \;\to\infty,\\
2\pi\alpha'\,\Theta^{r\theta}(r) &= \frac{1}{5\,r^{5/2}}\int_0^{2\pi} d\sigma\,\mu(\sigma)\,\gamma(\sigma)^4 \;\to\infty,\\
2\pi\alpha'\,\Theta^{\theta\phi}(r) &= \frac{1}{10\,r^{3}}\int_0^{2\pi} d\sigma\,\mu(\sigma)\,\nu(\sigma)\,|\gamma(\sigma)|^3 \;\to\infty \; .
\end{aligned} \tag{8.26}$$

Notice that the invariant string size tends to infinity [see eq.(8.19)] with $4\,a'(\sigma)^2$ as proportionality factor. Since $-2a(\sigma)$ is the leading behaviour of $t(\sigma,\tau)$, this suggests us that the string stretches infinitely in the (spatial) t direction when $r \to 0$.

As a matter of fact, infinitely growing string sizes are not observed in cosmological spacetimes [1,3] for strings exhibiting radiation (dual to unstable) behaviour.

For particular string solutions the energy-momentum tensor and the string size can be less singular than in the generic case. For ring solutions [12], $\mu(\sigma) = \mu =$ constant, $g(\sigma) = g =$ constant, $a(\sigma) = \nu(\sigma) = 0$, there is no stretching and

$$S = r \sin g \to 0$$

$$E = P_\theta = \frac{1}{\alpha'}\,\frac{\mu^5}{80\,r} \to +\infty \quad , \quad P_\phi = 0 \; .$$

There is no string stretching but the string keeps exhibiting dual to unstable behaviour. This is due to the balance of the tidal forces thanks to the special symmetry of the solution. It behaves in this special case as **one**-dimensional massless particles for $r \to 0$.

As is easy to see, setting $\mu(\sigma) = 0, g(\sigma) = \pi/2$ all equatorial string solutions behave as **one**-dimensional massless particles for $r \to 0$.

The resolution method used here for strings in black hole spacetimes is analogous to the expansions for $\tau \to 0$ developped in ref. [10,11] for strings in cosmological spacetimes (see sec. IV.A).

D. Axisymmetric Ring Solutions

We present in this section an axisymmetric ansatz describing a ring string in Schwarzschild spacetime. This ansatz has a symmetry compatible with the equations (8.5, 8.6) and (8.7) and hence separates them into ordinary differential equations [12].

$$\phi = n\sigma\,, \quad \theta = \theta(\tau)\,, \quad t = t(\tau)\,, \quad r = r(\tau)\,. \tag{8.27}$$

where $n =$ integer. This ansatz inserted in equations (8.5, 8.6) and (8.7) produces the following set of equations:

$$\begin{aligned} \ddot{r} - (r - 3/2)\,\dot{\theta}^2 + n^2 \sin^2\theta\,(r - 1/2) &= 0\,, \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} + n^2 r \sin\theta\cos\theta\, &= 0\,. \end{aligned} \tag{8.28}$$

with a conserved quantity

$$e^2 = \dot{r}^2 + r(r-1)(\dot{\theta}^2 + n^2\sin^2\theta)\,, \tag{8.29}$$

and t given by

$$\dot{t} = \frac{er}{r-1}\,. \tag{8.30}$$

Note that these equations are invariant under the change $\tau \to -\tau$, $t \to -t$. Since $\dot{t} > 0$ outside the horizon, $t(\tau)$ is a monotonous function, and we can use either τ or t to study the time evolution for $r > 1$.

The string energy is found to be in this case,

$$E(t) \equiv -P_0 = -\frac{G_{00}}{\alpha'}\frac{dX^0}{d\tau} = \frac{e}{\alpha'}\,.$$

where we used eq.(2.27):

$$P^0 = \int d^{D-1}X\sqrt{-G}\;T^{00}(X)\,.$$

The invariant size of the string in this case is

$$S = nr\sin\theta\,.$$

A useful equation, satisfied by solutions of these equations, is

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + n^2\right)(r\sin\theta) = \frac{1}{2}(n^2\sin^2\theta - 3\dot{\theta}^2)\sin\theta\,. \tag{8.31}$$

Let us now examine the possible asymptotic behaviours of these equations in different regimes. A first interesting question is the existence of collapsing solutions and the corresponding critical exponents. On computation, we find two possible collapsing behaviours from (8.28-8.29) , with the adequate choice of origin for τ for $\tau \to 0$:

$$\begin{aligned} r &\overset{\tau\to 0}{\simeq} \alpha\,\tau^{2/5}\,, \\ \theta &\overset{\tau\to 0}{\simeq} \theta_f + 2\,\sqrt{\alpha}\,\tau^{1/5}\,, \end{aligned} \tag{8.32}$$

where α is a constant and

$$r \overset{\tau\to 0}{\simeq} \; e\tau + \frac{n^2\sin^2\theta_f}{4}\,\tau^2 - \frac{n^2 e\sin^2\theta_f}{6}\,\tau^3 + O(\tau^4)\,,$$

$$\theta \overset{\tau\to 0}{\simeq} \theta_f - \frac{n^2 \sin(2\theta_f)}{12}\,\tau^2 + \frac{n^2 \sin^3\theta_f \cos\theta_f}{726}\,\tau^3 + O(\tau^4)\,. \tag{8.33}$$

This last one is obviously subdominant with respect to eq.(8.32) .

Consider now the regime given by large r and $\dot{\theta}^2/r$ small. From equations (8.29) and (8.31), we have that, for $|\tau| \to +\infty$,

$$\begin{aligned} r &\overset{\tau\to+\infty}{\simeq} p|\tau|\,, \\ \theta &\overset{\tau\to+\infty}{\simeq} \theta_0 - \frac{m}{p}\frac{\cos(n\tau+\varphi_0)}{\tau}\,, \end{aligned} \tag{8.34}$$

together with

$$e^2 = p^2 + n^2 m^2\,,$$

and $t \overset{\tau\to+\infty}{\sim} e\tau$. Here θ_0 is such that $\sin\theta_0 = 0$, i.e., $\theta = l\pi$ with l an integer. We could here understand p as an asymptotic radial momentum, e the energy, and nm the mass of the string. The latter is determined by the amplitude of the string oscillations.

We find for large τ that

$$\begin{aligned} x = r\sin\theta\cos\phi &= (-1)^{l+1}\; m\;\; \cos(n\tau+\varphi_0)\cos n\sigma\,, \\ y = r\sin\theta\sin\phi\, &= (-1)^{l+1}\; m\;\; \cos(n\tau+\varphi_0)\sin n\sigma\,. \end{aligned} \tag{8.35}$$

In this region, spacetime is minkowskian, and we can recognize (8.35) as the n^{th} excitation mode of a closed string. For $|n| = 1$ this corresponds at the quantum level to a graviton and/or a dilaton. Notice that m is the amplitude of the string oscillations.

The string size is here $S(\tau) = r(\tau)\;|\sin\theta(\tau)|$. We find from eqs.(8.32 -8.34) that

$$\begin{aligned} S(\tau) &\overset{\tau\to+\infty}{\simeq} m|\cos(\tau+\varphi)| \overset{\tau\to+\infty}{\sim} \frac{m}{\sqrt{2}} \\ S(\tau) &\overset{\tau\to+0}{\simeq} \alpha\sin\theta_f\;\tau^{2/5}\,. \end{aligned} \tag{8.36}$$

Whenever the string is not swallowed by the black hole, equation (8.34) describes both the incoming and outgoing regions $\tau \to \pm\infty$. However, the mass m, the momentum p and the phase φ_0 are in general different in the two asymptotic regions. This is an illustration of a rather general phenomenon noticed at the quantum level: particle transmutation [48]. This means that the excitation state of a string changes in general when it is scattered by an external field like a black hole. Within our classical ansatz (8.27), the only possible changes are in amplitude (mass), momentum, and phase. It can be seen numerically that the excitation state is indeed modified by the interaction with the black hole.

Due to the structure of our ansatz, the string, if it is not absorbed for some finite τ, may return to $z = +\infty$ ($\theta_f = 0 \pmod{2\pi}$), where it started at $\tau = -\infty$, or go past the black hole towards $z = -\infty$ ($\theta_f = \pi \pmod{2\pi}$)

An special case of interest is that of solutions such that $r(\tau) = r(-\tau)$. It follows that $\dot{\theta}^2(\tau) = \dot{\theta}^2(-\tau)$, from which $\theta(\tau) = \Delta - \epsilon\theta(-\tau)$, with ϵ a sign. We then see that Δ is restricted to multiples of π if the string does not fall into the black hole. If it is an odd multiple, we understand that the string has circled round the black hole a number of times and then has continued to infinity, whereas when it is an even multiple, the string bounces back after some dithering around the black hole. This analysis can be extended to all solutions.

Let us now analyze the absorption of the string by the black hole. If r starts at $+\infty$ for $\tau = -\infty$ and decreases ($\dot{r} < 0$), $\dot{r}$ must change sign at the periastron at time τ_0. Otherwise the singularity at $r = 0$ will be reached. Furthermore, $\ddot{r}(\tau_0) > 0$. We see from the first equation in the set (8.28) that this implies that $r(\tau_0) > 3/2$. In other words, if the string penetrates the $r < 3/2$ region, it will necessarily fall into the singularity. In yet another paraphrase, there is an effective horizon for ring string solutions. The surface $r = 3/2$ is necessarily contained within this horizon. Let us recall that for massless geodesics the effective horizon is a sphere of radius $r = \frac{3}{2}\sqrt{3}$.

To illustrate these points, we adjunct some figures. They depict the motion of ring-like strings described by equations (8.27) through (8.30). We numerically integrate equations (8.28) from large negative τ, where the asymptotic behaviour (8.34) holds. We choose θ_0, $n = 1$, and vary the values of p, m and φ_0. Depending on this last set of three values the string is absorbed or not by the black hole.

In figs. 8 we show an example of direct fall (i.e., with no bobbing around the black hole).

Figs. 8 : Numerical solution for the equations of motion of a string in a Schwarzschild black hole background: the string falls into the black hole.

In order to compare with this one, we next portray (figs. 9) a case where the string goes past the black hole before returning to it and collapsing. The clearest view of this event is given by fig. figbhpas d, which depicts $z = r\cos\theta$ as a function of $\rho = r\sin\theta$.

Figs. 9 : Numerical solution for the equations of motion of a string in a Schwarzschild black hole background: the string falls into the black hole, but only after first going past it and then back into the singularity.

Figs. 10 is dedicated to a non-falling string. It is particularly interesting to point out that the excitation state has been changed by scattering by the black hole, as can be clearly seen from the third graph in this figure, which depicts $r\sin\theta$ as a function of τ. We see that the oscillation amplitude is larger after the collision than before. This means that the outgoing string mass is larger than the ingoing string mass. Hence, particle transmutation in the sense of ref. [48] takes place here.

In the fourth of this series, fig. 10 d, we portray $z = r\cos\theta$ against $\rho = r\sin\theta$. It is to be remarked that the string **bounces** (the lower end of the picture), then oscillates around the black hole, and finally escapes to infinity.

Figs. 10 : Numerical solution for the equations of motion of a string in a Schwarzschild black hole background: the string goes past the black hole, circles round it, and then bounces back with a change in its amplitude and momentum.

That the string be absorbed or not by the black hole is dictated by whether it comes or not within the effective horizon, as mentioned above. This, in turn, is crucially dependent on the phase φ_0 chosen as part of the initial data ($\tau \to -\infty$). Whatever value the mass (amplitude) m and the momentum p take, there is always some interval of values of φ_0 for which the string will be absorbed by the black hole.

Besides numerical experiments, this behaviour follows from the simple fact that a change of the initial phase φ_0 would displace the string worldsheet, thus possibly bringing it closer to the black hole.

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Figure Captions:

Figure 1: Plot of the function $HT(\tau)$, for two values of σ, for $n = 4, |x^0 >= (1 + i, .6 + .4i, .3 + .5i, .77 + .79i)$. The function $\tau(T)$ is multivalued, revealing the presence of five strings.

Figure 2: Same as fig.1, for $n = 4, |x^0 >= (1, -1, i, 1)$. Because of a degeneracy, there are now only three strings.

Figure 3: $\tau = \tau(\sigma, T)$ for fixed T for $n = 4, |x^0 >= (1, -1, i, 1)$. Three values of HT are displayed, corresponding to HT=0 (full line), 1 (dots), and 2 (dashed line). For each HT, three curves are plotted, which correspond to the three strings. They are ordered with τ increasing.

Figure 4: Same as fig. 3 for $n = 4, x^0 >= (1 + i, .6 + .4i, .3 + .5i, .77 + .79i)$. a) The five curves corresponding to the five strings at HT=2. b) The five curves for three values of HT: HT=0 (full line), 1 (dots), and 2 (dashed line).

Figure 5: Evolution as a function of cosmic time HT of the three strings, in the comoving coordinates (X^1, X^2), for $n = 4, |x^0 >= (1, -1, i, 1)$. The comoving size of string (1) stays constant for $HT < -3$, then decreases around $HT = 0$, and stays constant again after $HT = 1$. The invariant size of string (2) is constant for negative HT, then grows as the expansion factor for $HT > 1$, and becomes identical to string (1). The string (3) has a constant comoving size for $HT < -3$, then collapses as e^{-HT} for positive HT.

Figure 6: Evolution of three of the five strings for $n = 4, |x^0 >= (1+i, .6+.4i, .3+.5i, .77+ .79i)$.

Figure 7: Evolution of the three strings for the degenerate case $n = 6, |x^0 >= (1, -1, i, 1)$.

Note:
Figures 1 to 7 can be find in ref. [6].
Figures 8 to 10 can be find in ref. [12]